

M1 Health-AI Combinatorics - Counting

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Combinatorics is a branch of mathematics that studies the different ways of counting, organizing, and selecting objects according to certain rules. In particular, it aims to determine how many possibilities exist in a given situation without having to list them all. It is used in many fields, such as

- probability

- computer science

- cryptography

- graph theory

- optimization.

For example, combinatorics can be used to calculate the number of ways to choose or arrange several elements.

- Induction
- Recurrence
- Generating Functions

We want to prove that a statement, depending on a natural number n , is true for any n .

A very powerful tool in mathematics is the *method of mathematical induction*:

- **The initial step:** Prove that the statement is true for the smallest value of n for which it is defined ($n = 0$, or $n = 1$)
- **The induction step:** prove that from the fact that the statement is true for n (the *induction hypothesis*), it follows that the statement is also true for $n + 1$.

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Example. For all positive integer n , we have

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Example. For all positive integer n , the number of subsets of $[n]$ is 2^n .

Example. Let $a_0 = 1$, and let $a_{n+1} = 3a_n + 1$, for all $n \geq 1$. Find an explicit formula for a_n .

Example. For all positive integer n , and all $r \neq 1$, we have

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

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Strong induction:

- **The initial step:** Prove that the statement is true for the smallest value of n for which it is defined ($n = 0$, or $n = 1$)
- **The induction step:** Prove that from the fact that the statement is true for all integers less than $n + 1$ ('the induction hypothesis'), it follows that the statement is also true for $n + 1$.

Example. Let the sequence (a_n) be defined by the relation $a_0 = 1$, and $a_{n+1} = a_0 + a_1 + \cdots + a_n + 1$, for $n \geq 0$. Prove, for all positive integer n , that we have $a_n = 2^n$.

Recurrence relations

Let T_n be the set of completely covers (or tilings) of a n rectangle with $2 \cdot 1$ dominoes.

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Example

$$T_1 = \boxed{}$$

$$T_2 = \boxed{} \boxed{}, \boxed{} \boxed{}$$

$$T_3 = \boxed{} \boxed{} \boxed{}, \boxed{} \boxed{} \boxed{}, \boxed{} \boxed{} \boxed{}$$

$$T_4 = \boxed{} \boxed{} \boxed{} \boxed{}, \boxed{} \boxed{} \boxed{} \boxed{}, \boxed{} \boxed{} \boxed{} \boxed{}, \boxed{} \boxed{} \boxed{} \boxed{}, \boxed{} \boxed{} \boxed{} \boxed{}$$

Can we find a recursive definition for T_n ?

What is $f_n = \text{card}(T_n)$?

f_n is the Fibonacci number sequence

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^{n+1} - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^{n+1} .$$

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Generating functions

A cashier needs to give a customer 19 cents in change. She has 20 identical one-cent coins and 2 identical five-cent coins.

- In how many ways can she give the exact change?
- Same question if she has an unlimited number of one-cent and five-cent coins.
- Same question if she also has an unlimited number of ten-cent coins.

Definition Let $a_0, a_1, \dots, a_k, \dots$ be a sequence of real numbers. Then the formal power series

$$A(z) = \sum_{k \geq 0} a_k z^k$$

is called the *ordinary generating function* of the sequence $a_0, a_1, \dots, a_k, \dots$

The coefficient of z^k is denoted by $[z^k]A(z)$ and it is exactly a_k .

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Example $a_n = 1, n \geq 0$

$$\begin{aligned} A(z) &= \sum_{n \geq 0} a_n \cdot z^n \\ &= \sum_{n \geq 0} z^n \\ &= \frac{1}{1 - z} \end{aligned}$$

Example $a_n = c^n, n \geq 0$

$$\begin{aligned} A(z) &= \sum_{n \geq 0} a_n \cdot z^n \\ &= \sum_{n \geq 0} c^n \cdot z^n \\ &= \sum_{n \geq 0} (c \cdot z)^n \\ &= \frac{1}{1 - c \cdot z} \end{aligned}$$

sequence	power series	g. function
$1, 1, 1, 1, \dots, 1, \dots$	$\sum_{n \geq 0} z^n$	$\frac{1}{1-z}$
$0, 1, 2, 3, \dots, n, \dots$	$\sum_{n \geq 0} n z^n$	$\frac{z}{(1-z)^2}$
$1, 2, 3, 4, \dots, (n+1), \dots$	$\sum_{n \geq 0} (n+1) z^n$	$\frac{1}{(1-z)^2}$
$1, 2, 4, 8, \dots, 2^n, \dots$	$\sum_{n \geq 0} 2^n z^n$	$\frac{1}{1-2z}$
$1, c, c^2, c^3, \dots, c^n, \dots$	$\sum_{n \geq 0} c^n z^n$	$\frac{1}{1-cz}$
$0, 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots$	$\sum_{n \geq 0} \frac{1}{n} z^n$	$\ln \frac{1}{1-z}$
$0, 0, 1, 3, 6, 10, \dots, \binom{n}{2}, \dots$	$\sum_{n \geq 2} \binom{n}{2} z^n$	$\frac{z^2}{(1-z)^3}$

Example The frog population of a lake grows fourfold each year. On the first day of each year, 100 frogs are taken out of the lake and shipped into another lake. Assuming that there were 50 frogs in the lake originally, how many frogs will be there in 20 years? In n years?

We get $a_0 = 50$ and

$$a_n = 4 \cdot a_{n-1} - 100$$

if $n \geq 1$.

What is

$$A(z) = \sum_{n \geq 0} a_n \cdot z^n?$$

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$$A(z) = \sum_{n \geq 0} a_n \cdot z^n?$$

Recipe

- Take a copy of this relation for each integer $n \geq 1$
- multiply both sides by z^n , and
- take the sum of the infinity many equations obtained

$$\sum_{n \geq 1} a_n \cdot z^n = 4 \cdot \sum_{n \geq 1} a_{n-1} \cdot z^n - \sum_{n \geq 1} 100 \cdot z^n$$

$$\sum_{n \geq 0} a_n \cdot z^n - a_0 = 4z \cdot \sum_{n \geq 1} a_{n-1} \cdot z^{n-1} - 100 \cdot (\sum_{n \geq 0} 100 \cdot z^n - 1)$$

$$A(z) - a_0 = 4 \cdot z \cdot A(z) - \frac{100z}{1-z}$$

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$$\sum_{n \geq 0} a_n \cdot z^n - a_0 = 4z \cdot \sum_{n \geq 1} a_{n-1} \cdot z^{n-1} - 100 \cdot (\sum_{n \geq 0} 100 \cdot z^n - 1)$$

$$A(z) - a_0 = 4 \cdot z \cdot A(z) - \frac{100z}{1-z}$$

We compressed the information given by the infinity many equations of type $a_n = 4a_{n-1} - 100$ into just one equation.

$$A(z) = \frac{50}{1-4z} - \frac{100z}{(1-z)(1-4z)}$$

The coefficient of z^n on the right-hand side of this relation is

$$\begin{aligned} a_n &= 50 \cdot 4^n - 100 \cdot \frac{4^n - 1}{3} \\ &= 50 \cdot \frac{4^n}{3} + \frac{100}{3} \end{aligned}$$

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Let us summarize the technique we have just learned to turn recursive formulae into explicit ones

- 1 Define the generating function $G(z)$ of the sequence $(a_n)_{n \geq 0}$
- 2 Transform the recursive formula into an equation in $G(z)$. This can usually be done by multiplying both sides of the recursion by z^n and summing for n
- 3 Solve for $G(z)$
- 4 Find the coefficient of z^n in $G(z)$. As this is a_n , this will provide an explicit formula for a_n .

Example

$$f_n = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ f_{n-1} + f_{n-2} & \text{if } n > 1. \end{cases}$$

1. Define the generating function $F(z)$ of the sequence $(f_n)_{n \geq 0}$

$$F(z) = \sum_{n \geq 0} f_n \cdot z^n$$

2. Transform the recursive formula into an equation in $F(z)$.

$$\sum_{n \geq 2} f_n \cdot z^n = \sum_{n \geq 2} f_{n-1} \cdot z^n + \sum_{n \geq 2} f_{n-2} \cdot z^n$$

$$F(z) - z = zF(z) + z^2F(z)$$

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3. Solve for $F(z)$

$$F(z) = \frac{z}{1 - z - z^2} = \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \phi z} - \frac{1}{1 - \psi z} \right)$$

with $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$

4. Find the coefficient of z^n in $F(z)$.

$$f_n = \frac{1}{\sqrt{5}} (\phi^n - \psi^n)$$

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Operations on generating functions

$$A(z) = \sum_{k \geq 0} a_k z^k, B(z) = \sum_{k \geq 0} b_k z^k$$

Sum

$A(z) + B(z)$ is the generating function of

$$a_0 + b_0, a_1 + b_1, a_2 + b_2, \dots$$

Right shift

$$zA(z) = \sum_{n \geq 1} a_{n-1} z^n$$

is the generating function of the sequence

$$0, a_0, a_1, a_2, \dots, a_{n-1}, \dots$$

Example The sequence $0, 1, 1, 1, \dots$ has the generating function

$$A(z) = \frac{z}{1-z}$$

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Example What is the g.f. of the sequence $2, 4, \dots, 2^{n+1} \dots$

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Example What is the g.f. of the sequence $2, 4, \dots, 2^{n+1} \dots$

Differentiation

$$A'(z) = \sum_{n \geq 0} (n+1) a_{n+1} z^n$$

is the generating function of the sequence

$$a_1, 2a_2, 3a_3, 4a_4, \dots, (n+1)a_{n+1}, \dots$$

Example The generating function of the sequence $1, 2, 3, \dots, (n+1), \dots$ that is, $a_n = n+1$, is

$$F(z) = \frac{1}{(1-z)^2}$$

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Integration

$$\int_0^z A(t) dt = \sum_{n \geq 1} \frac{a_{n-1}}{n} z^n$$

is the generating function of the sequence

$$0, a_0, \frac{a_1}{2}, \frac{a_2}{3}, \frac{a_3}{4}, \frac{a_4}{5}, \dots, \frac{a_{n-1}}{n}$$

Example $0, 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots$
that is, $a_n = \frac{1}{n}$, is

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Re-scaling

$$A(\lambda z) = \sum_{n \geq 0} \lambda^n a_n z^n$$

is the generating function of the sequence
 $a_0, \lambda a_1, \lambda^2 a_2, \dots, \lambda^n a_n, \dots$

Example

$$1, \frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{2^n}, \dots$$

$$a_n = \frac{1}{2^n}$$

$$A(z) = \frac{1}{1 - z/2}$$

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$$a_n = \frac{1}{2^n}$$

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Difference

$$(1 - z)A(z) = a_0 + \sum_{n \geq 1} (a_n - a_{n-1})z^n$$

is the generating function of the sequence

$a_0, a_1 - a_0, \dots, a_n - a_{n-1}, \dots$

Example

$1, -\frac{1}{2}, +\frac{1}{4}, -\frac{1}{8}, \dots, \dots$

$a_n = \left(-\frac{1}{2}\right)^n$

Difference

$$(1 - z)A(z) = a_0 + \sum_{n \geq 1} (a_n - a_{n-1})z^n$$

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Example

$$1, -\frac{1}{2}, +\frac{1}{4}, -\frac{1}{8}, \dots, \dots$$

$$a_n = \left(-\frac{1}{2}\right)^n$$

Convolution

$$A(z)B(z) = \sum_{n \geq 0} \left(\sum_{0 \leq k \leq n} a_k b_{n-k} \right) z^n$$

is the generating function of the sequence

$$a_0 b_0, a_1 b_0 + a_0 b_1, \dots, \sum_{0 \leq k \leq n} a_k b_{n-k}, \dots$$

Partial sum

$$\frac{A(z)}{1-z} = \sum_{n \geq 0} \left(\sum_{0 \leq k \leq n} a_k \right) z^n$$

is the generating function of the sequence

$$a_0, a_0 + a_1, a_0 + a_1 + a_2, \dots, \sum_{0 \leq k \leq n} a_k, \dots$$

Example $0, 1, 3, 6, \dots, \frac{n(n+1)}{2}, \dots$

$$a_n = \frac{n(n+1)}{2}$$

$$A(z) = \frac{z}{(1-z)^3}$$

Partial sum

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The product formula

Theorem

- a_n = the number of ways to build a certain structure on an n -element set, and $A(z)$ its generating function
- b_n = the number of ways to build another structure on an n -element set, and $B(z)$ its generating function
- c_n = the number of ways to separate $\{1, 2, \dots, n\}$ into intervals

$$S = \{1, 2, \dots, i\}, \text{ and}$$

$$T = \{i + 1, i + 2, \dots, n\}$$

(the intervals S and T are allowed to be empty)
and then to build a structure of the first kind on S and a structure of second kind on T . Let $C(z)$ the generating function of $(c_n)_{n \geq 0}$

$$C(z) = A(z) \cdot B(z)$$

Example The semester of the UBE consists of n days. At the beginning of each semester, the Dean designs the term in the following ways. He splits the term into two parts, and chooses some days for independent study in both parts of the semester. In how many different ways can he plan the semester with these constraints?

Proof

$a_n = b_n = 2^n$ with the generating function $A(z) = B(z) = \frac{1}{1-2z}$.

So, $C(z) = A(z) \cdot A(z) = \frac{1}{(1-2z)^2}$

Remark that $C(z) = \frac{1}{2}A'(z)$, and so $c_n = (n+1) \cdot 2^n$

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Theorem Let

- a_n the number of ways to build a certain structure on an n element set, and assume $a_0 = 0$. Denote

$$A(z) = \sum_{n \geq 0} a_n z^n$$

- h_n the number of ways to split the set $\{1, 2, \dots, n\}$ into an unspecified number of distinct nonempty intervals, then build a structure of the given kind on each of these intervals, assume $h_0 = 1$. Denote h_n the number of ways we can do this and

$$H(z) = \sum_{n \geq 0} h_n z^n$$

We have

$$H(z) = 1 + A(z) + A(z)^2 + A(z)^3 \dots = \frac{1}{1 - A(z)}$$

Example All n soldiers of a military squadron stood in a line. The officers in charge splits the line at several place, forming smaller (nonempty) units. Then he names one person in each unit to be commander of that unit. Let h_n be the number of ways he can do this. Find a closed formula for h_n .

Proof Let h_n the number of ways the officer in charge can proceed. Let a_k be the number of ways to choose a commander from a unit of k people.

Clearly, $a_k = k$, and

$$A(z) = \sum_{k \geq 1} k \cdot x^k = \frac{x}{(1-x)^2}$$

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From the previous theorem,

$$\begin{aligned}H(z) &= \frac{1}{1 - A(z)} \\&= \frac{1}{1 - \frac{x}{(1-x)^2}} \\&= 1 + \frac{x}{1 - 3x + x^2} \\&= 1 + x \cdot \frac{1}{\sqrt{5}} \left(\frac{1}{x - \alpha} - \frac{1}{x - \beta} \right) \\&= 1 + x \cdot \frac{1}{\sqrt{5}} \left(\frac{\alpha}{1 - \alpha x} - \frac{\beta}{1 - \beta x} \right)\end{aligned}$$

with $\alpha = \frac{3+\sqrt{5}}{2}$ and $\beta = \frac{3-\sqrt{5}}{2}$
and finally

$$h_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$$

The compositional formula

Theorem Let

- a_n the number of ways to build a certain structure on an n element set,
 - assume $a_0 = 0$
 - denote $A(z) = \sum_{n \geq 0} a_n z^n$
- b_n the number of ways to build a second structure on an n element set,
 - assume $b_0 = 0$
 - denote $B(z) = \sum_{n \geq 0} b_n z^n$

- g_n the number of ways to
 - split the set $\{1, 2, \dots, n\}$ into an unspecified number of distinct nonempty intervals,
 - then build a structure of the first kind on each of these intervals, and
 - then to build a structure of the second kind on the set of intervals, set $g_0 = 1$.

Denote $G(z)$ the generating function of the sequence g_n .

We have

$$G(z) = B(A(z))$$

Example All n soldiers of a military squadron stand in a line. The officers in charge splits the line at several place, forming smaller (nonempty) units. Then he chooses a (possibly empty) subset of the newly formed units for night duty. In how many different ways can he do this?

Proof

$a_n = 1$ for all $n \geq 1$, as there is one way to put the trivial structure

(that is to say, no structure as all) on the individual units. So,

$$A(z) = \frac{z}{1-z}$$

$b_n = 2^n$ as we simply choose a subset of the set of all intervals. Therefore,

$$B(z) = \frac{1}{1-2z}$$

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So,

$$\begin{aligned} G(z) &= B(A(z)) \\ &= \frac{1}{1 - \frac{2z}{1-z}} \\ &= \frac{1-z}{1-3z} \\ &= \frac{1}{1-3z} - \frac{z}{1-3z} \end{aligned}$$

And finally $g_n = 2 \cdot 3^{n-1}$ for $n \geq 1$.