The equidistribution of some descent set based statistics on words

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Outline

- An alternative proof of the equidistribution of (Des, Ides) and (Desrc, Ides) on permutations
- Generalization of this equidistribution to *the rearrangement of the letters of a word*
- Some (more) definitions

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- An involution p on words :

$$(\mathrm{f}, \mathrm{des}, \mathrm{maj}, \mathrm{stat}, \mathrm{ides})\, u = (\mathrm{f}, \mathrm{des}, \mathrm{stat}, \mathrm{maj}, \mathrm{ides})\, p(u)$$

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- An involution m on *the rearrangement of the letters of a word* :

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Simultaneously and independently, similar results have been obtained :

MAHONIAN STAT ON REARRANGEMENT CLASS OF WORDS

SHISHUO FU AND TING HUA

Some definitions

Des u is the descent set of u and $\operatorname{des} u = \operatorname{card}(\operatorname{Des} u)$

$\mathrm{r}(u)$, the *reverse* of u, is $u_n u_{n-1} \dots u_1$,

$\mathrm{c}(u)$, the *complement* of u, is
$(q - u_1 + 1)(q - u_2 + 1) \dots (q - u_n + 1)$,

$\mathrm{rc}(u)$, the *reverse-complement* of u,

$$\mathrm{r}(\mathrm{c}(u)) = \mathrm{c}(\mathrm{r}(u))$$

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- $\pi \in S_n$
  $\mathrm{Ides}\,\pi = \mathrm{Des}\,\pi^{-1}$, or equivalently

  $\mathrm{Ides}\,\pi = \{a : \text{there exist } i < j \text{ with } \pi_i = a+1 \text{ and } \pi_j = a\}.$

- For a word u
  and an a different from the largest symbol in u, $next_u(a)$ denotes the smallest symbol in u, larger than a. A natural generalization of Ides to words is given by :

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Theorem 2. *There exists an involution* $\boldsymbol{j}$ *of* S *preserving* IDOWN *and exchanging* DOWN *with its complement to* n. *In other words one has identically*

$$\mathrm{IDOWN}\, \boldsymbol{j}s = \mathrm{IDOWN}\, s$$

and

$$\mathrm{DOWN}\, \boldsymbol{j}s = \{n - x : x \in \mathrm{DOWN}\, s\}\ .$$

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Theorem

There is an involution $\phi : S_n \to S_n$ *such that*

$$(\mathrm{Des}, \mathrm{Ides})\, \phi(\pi) = (\mathrm{Desrc}, \mathrm{Ides})\, \pi.$$

Permutations expansion

For $\pi \in S_n$ and $0 \leq i \leq n$ the ith expansion of π, denoted by $\epsilon(i, \pi)$, is the length $n+1$ permutation

$$(\pi_1 + 1)(\pi_2 + 1) \dots (\pi_i + 1)1(\pi_{i+1} + 1) \dots (\pi_n + 1).$$

We extend ϵ to a sequence of expansions : $\epsilon(i_1, i_2, \ldots, i_k; \pi)$ is the permutation obtained by applying the i_1th expansion to π, then the i_2th expansion to the obtained permutation, ...

Proposition

- *Any $\pi \in S_n$ can be obtained uniquely from $1 \in S_1$ by a sequence of expansions*

  $$i_1, i_2, \ldots, i_{n-1} \in \{0,1\} \times \{0,1,2\} \times \cdots \times \{0,1,\ldots,n-1\} = SE_n$$

- *If $\pi = \epsilon(i_1, i_2, \ldots, i_{n-1}; 1) \in S_n$, then*

  $$i_k < i_{k+1} \text{ iff } \pi_{n-k} \in \mathrm{Ides}\,\pi$$

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Definition

The descent vector of $\pi \in S_n$ is the binary word $b_1 b_2 \dots b_{n-1}$ with $b_i = 1$ iff i is a descent in π.

Binary word expansion

The ith expansion of $b \in \{0,1\}^n$, $0 \leq i \leq n+1$, denoted $\epsilon(i;b)$, is defined as :

$$\epsilon(i;b) = \begin{cases} 0b_1 b_2 \dots b_n & \text{if } i = 0, \\ b_1 \dots b_{i-1} 10 b_{i+1} \dots b_n & \text{if } 1 \leq i \leq n, \\ b_1 b_2 \dots b_n 1 & \text{if } i = n+1, \end{cases}$$

and in particular, the expansions of the empty word λ are $\epsilon(0;\lambda) = 0$ and $\epsilon(1;\lambda) = 1$.

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Proposition

If $b \in \{0,1\}^n$, then there is a permutation $\rho = \rho_b$ of the set $\{0,1,\ldots,n+1\}$ such that

$$\mathrm{r}(\epsilon(i;b)) = \epsilon(\rho(i);\mathrm{r}(b)).$$

Remark

$\rho_{\mathrm{r}(b)} = \rho_b^{-1}$.

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Example

$$\lambda \xrightarrow{0} 0 \xrightarrow{0} 00 \xrightarrow{3} 001 \xrightarrow{2} 0101 = \epsilon(0, 0, 3, 2; \lambda)$$

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Any binary word b can be obtained (not necessarily in a unique way) by iteratively expanding the empty word λ, that is, $b = \epsilon(i_1, i_2, \ldots, i_k; \lambda)$ for appropriate integers $i_1, i_2, \ldots, i_k$ where k is the length of b ; and b is uniquely determined by the sequence $i_1, i_2, \ldots, i_k$.

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$$\lambda \xrightarrow{0} 0 \xrightarrow{0} 00 \xrightarrow{3} 001 \xrightarrow{2} 0101 = \epsilon(0, 0, 3, 2; \lambda)$$

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Proposition

Let $a \in \{0,1\}^n$, $b = \mathrm{r}(a)$ and $k \geq 1$.

1. *For any k increasing integers $i_1, i_2, \ldots, i_k$ in $\{0, 1, \ldots, n+k\}$ there are other k increasing integers $j_1, j_2, \ldots, j_k$ in $\{0, 1, \ldots, n+k\}$ such that $c = \epsilon(i_1, i_2, \ldots, i_k; a)$ and $d = \epsilon(j_1, j_2, \ldots, j_k; b)$ are related by $d = \mathrm{r}(c)$.*
2. *For any k weakly decreasing integers $i_1, i_2, \ldots, i_k$ in $\{0, 1, \ldots, n+1\}$ there are other k weakly decreasing integers $j_1, j_2, \ldots, j_k$ in $\{0, 1, \ldots, n+1\}$ such that $c = \epsilon(i_1, i_2, \ldots, i_k; a)$ and $d = \epsilon(j_1, j_2, \ldots, j_k; b)$ are related by $d = \mathrm{r}(c)$.*

Proposition

Let $a \in \{0,1\}^n$, $b = \mathrm{r}(a)$ *and* $k \geq 1$.

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2. *For any* k *weakly decreasing integers* $i_1, i_2, \ldots, i_k$ *in* $\{0,1,\ldots,n+1\}$ *there are other* k *weakly decreasing integers* $j_1, j_2, \ldots, j_k$ *in* $\{0,1,\ldots,n+1\}$ *such that* $c = \epsilon(i_1, i_2, \ldots, i_k; a)$ *and* $d = \epsilon(j_1, j_2, \ldots, j_k; b)$ *are related by* $d = \mathrm{r}(c)$.

For a given binary word $a \in \{0,1\}^n$, we denote by $R_a(i_1, i_2, \ldots, i_k)$ the sequence $j_1, j_2, \ldots, j_k$. In particular, when $k = 1$, we have $R_a(i) = \rho_a(i)$.

Remark

If for a binary word a we have $R_a(i_1, i_2, \ldots, i_k) = j_1, j_2, \ldots, j_k$, then $R_b(j_1, j_2, \ldots, j_k) = i_1, i_2, \ldots, i_k$ for $b = \mathrm{r}(a)$.

Let

$$I = i_1, i_2, \ldots, i_{n-1} \in \{0,1\} \times \{0,1,2\} \times \cdots \times \{0,1,\ldots,n-1\},$$

and

$$I = I_1, I_2, \ldots, I_\ell$$

be the partition of $i_1, i_2, \ldots, i_{n-1}$ such that

- $I_1 = i_1, i_2, \ldots, i_{p_1}$,
  $I_2 = i_{p_1+1} i_{p_1+2} \ldots, i_{p_2}$
  $\ldots$
  $I_\ell = i_{p_{\ell-1}+1}, i_{p_{\ell-1}+2}, \ldots, i_{p_\ell}$
- I_k is a weakly decreasing subsequence of I if k is odd,
  I_k is a length maximal inreasing subsequence of I if k is even.

Theorem

There is an involution χ *on* SE_n *such that* $\chi(I_1, I_2 \ldots, I_\ell) = J_1, J_2 \ldots, J_\ell$ *with*

- *the length of* I_k *is equal to that of* J_k, $1 \leq k \leq \ell$
- *the binary word* $a = \epsilon(I_1, I_2 \ldots, I_\ell; \lambda)$ *and* $b = \epsilon(J_1, J_2 \ldots, J_\ell; \lambda)$ *are related by* $b = \mathrm{r}(a)$.

Theorem

There is an involution ϕ on S_n such that

$$(\mathrm{Des}, \mathrm{Ides})\,\phi(\pi) = (\mathrm{Desrc}, \mathrm{Ides})\,\pi.$$

Proof

If $\pi \in S_n$, then $\sigma = \epsilon(\chi(\epsilon^{-1}(\pi))) \in S_n$, and

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The equidistribution of (Des, Ides) and (Desrc, Ides) on the rearrangement of the letters of a word

For a length n word u the *expansion* of u, denoted $\exp(u)$, is the unique length n permutation π with $\pi_i < \pi_j$ if and only if either

- $u_i < u_j$, or
- $u_i = u_j$ and $i < j$.

In particular, if u is a permutation, then $\exp(u) = u$.

Example

$\exp(12112) = 14235$ and $\exp(22111) = 45123$.

For $a_1, a_2, \ldots, a_k$ positive integers, A denotes be the set of permutations of $\{1^{a_1}, 2^{a_2}, \ldots, k^{a_k}\}$
$\exp : A \to S_n$, with $n = a_1 + a_2 + \ldots + a_k$ can be defined as $\exp(u)$ is the permutation $\pi = \pi_1 \pi_2 \ldots \pi_n$ where

$$\pi_i = a_1 + a_2 + \ldots + a_{u_i - 1} + |u_1 u_2 \ldots u_{i-1}|_{u_i} + 1.$$

If we restrict the codomain of $\exp$ to the set of length n permutations having its Ides set included in $\{a_1, a_1 + a_2, \ldots, a_1 + a_2 + \ldots + a_{k-1}\}$, then $\exp$ becomes a bijection.
For a word u $\mathrm{Ides}\, u = \mathrm{Ides}\, \phi \circ \exp(u)$ and

$$\exp^{-1} \circ \phi \circ \exp : A \to A$$

is well defined and we still denote this mapping by ϕ.

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Theorem

For any word u we have

- $\phi(\phi(u)) = u$ *(that is ϕ is an involution),*
- $\phi(u)$ *is a rearrangement of the letters of u, and*
- $(\mathrm{Des}, \mathrm{Ides})\, u = (\mathrm{Desrc}, \mathrm{Ides})\, \phi(u)$.

Some (more) definitions

- 1231 has three occurrences as a pattern in 6214562 : 2452, 2462 and 2562.
- 213 has four occurrences as a pattern in 425163 : 425, 416, 216 and 516
- [213 has two occurrences in the permutation 425163 : 425 and 416
- in the word 6214562
  - 645 is an occurrence of 312
  - 262 is an occurrence of 121]
- f $u = [21\, u + 1$
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$$\operatorname{maj} u = \sum_{i \in \operatorname{Des} u} i$$

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An involution p on words :
$(\mathrm{f}, \mathrm{des}, \mathrm{maj}, \mathrm{stat}, \mathrm{ides})\, u = (\mathrm{f}, \mathrm{des}, \mathrm{stat}, \mathrm{maj}, \mathrm{ides})\, p(u)$

FPSAC 2010, San Francisco, USA DMTCS proc. **AN**, 2010, 601–612

On joint distribution of adjacencies, descents and some Mahonian statistics

Alexander Burstein[1]

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Abstract. We prove several conjectures of Eriksen regarding the joint distribution on permutations of the number of adjacencies (descents with consecutive values in consecutive positions), descents and some Mahonian statistics. We also prove Eriksen's conjecture that a certain bistatistic on Viennot's alternative tableaux is Euler-Mahonian.

Theorem 2.1

$$(\mathrm{f}, \mathrm{adj}, \mathrm{des}, \mathrm{maj}, \mathrm{stat})\, \pi = (\mathrm{f}, \mathrm{adj}, \mathrm{des}, \mathrm{stat}, \mathrm{maj})\, p(\pi).$$

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(f, adj, des, ides, maj, stat) u = (f, adj, des, ides, f, stat, maj) $p(u)$.
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(but not on the rearrangement class of an arbitrary word).

Theorem 2

We define an involution p on words.

We define two transformations :

$$u \mapsto u' \mapsto v$$

- $u \mapsto u'$

For the integers a, b and x, with $a \leq x \leq b$, the *complement* of x with respect to the interval $\{a, a+1, \ldots, b\}$ is $b - (x - a)$.
Let $u = u_1 u_2 \ldots u_n \in [q]^n$ with $u \notin [q-1]^n$ (i.e., the largest symbol of u is q). Let also $c = u_1 + 1$ if the symbol u_1 occurs once in u, and $c = u_1$ otherwise.
$u' \in [q]^n$ by $u'_1 = u_1$, and for $i \geq 2$, u'_i is the complement of u_i with respect to

- $\{c, c+1, \ldots, q\}$, if $u_i \geq u_1$, and
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$v \in [q]^n$ by $v_1 = u'_1 = u_1$, and $v_i = u'_{n-i+2}$ for $i \geq 2$.
Equivalently, $v_2 v_3 \dots v_n$ is the reverse of $u'_2 u'_3 \dots u'_n$.
Finally, p is the mapping $u \mapsto v$, and p is an involution on $[q]^n$. If u is a permutation, then so is $p(u)$

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Lemma

If $u \in \{1,2\}^n$ *with* $u_1 = 1$, *then*

$$(\mathrm{f}, \mathrm{des}, \mathrm{stat}, \mathrm{maj})\, u = (\mathrm{f}, \mathrm{des}, \mathrm{maj}, \mathrm{stat})\, p(u).$$

Proof

Let $v = p(u)$. Thus, $(\mathrm{f}, \mathrm{des})\, u = (\mathrm{f}, \mathrm{des})\, v$. In addition

$$\begin{aligned}
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If $u \in \{1,2\}^n$ *with* $u_1 = 2$, *then*

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If $u \in \{1,2\}^n$, *then*

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For u we denote by $\mathrm{flat}(u)$ the same length word u'

- if $u_1 = 1$, then $u' = 11\dots1$;
- othewise, $u'_i = \begin{cases} 1 & \text{if} \quad u_i < u_1, \\ 2 & \text{if} \quad u_i \geq u_1. \end{cases}$

Fact

For any word u, $p(\mathrm{flat}(u)) = \mathrm{flat}(p(u))$.

Corollary

If $u \in [q]^n$, $q \geq 2$, $n \geq 1$, then

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Theorem

If $u \in [q]^n$, *then*

$$(\mathrm{f}, \mathrm{des}, \mathrm{stat}, \mathrm{maj}, \mathrm{ides})\, u = (\mathrm{f}, \mathrm{des}, \mathrm{maj}, \mathrm{stat}, \mathrm{ides})\, p(u).$$

Proof

Let D_1 be the set of descents i in u with $u_i \geq u_1 > u_{i+1}$, and D_2 be the set of those with $u_i, u_{i+1} \geq u_1$ or $u_i, u_{i+1} < u_1$, and $v = p(u)$.

$$\begin{aligned}
\operatorname{stat} v &= \operatorname{comaj} v - [21\, v + \operatorname{des} v \\
&= \sum_{i \in \operatorname{Des} v} (n-i) - [21\, v + \operatorname{des} v \\
&= \sum_{i \in E_1} (n-i) - [21\, v + |E_1| + \sum_{i \in E_2} (n-i) + |E_2| \\
&= \operatorname{stat} \operatorname{flat}(v) + \sum_{i \in E_2} (n-i+1) \\
&= \operatorname{maj} \operatorname{flat}(u) + \sum_{i \in D_2} i \\
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Remark

p does not preserve the underlying multiset permutation, nor Ides statistic.

Rephrasing the definition of p

Let $u \in [q]^n$, s be the length $n-1$ suffix of u (that is, $s = s_1 s_2, \ldots s_{n-1} = u_2 u_3 \ldots u_n$), and $c = u_1 + 1$ if the symbol u_1 occurs once in u, and $c = u_1$ otherwise.

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Now $p(u)$ is obtained by applying the next three steps.

(1) Let $\{i_1, i_2, \ldots, i_a\}$ be the set of positions in s with $s_i < u_1$, and $\{j_1, j_2, \ldots, j_b\}$ those with $s_j \geq u_1$.

In other words, $s_{i_1} s_{i_2} \ldots s_{i_a}$ (resp., $s_{j_1} s_{j_2} \ldots s_{j_b}$) is the possibly scatterd subword of s formed by symbols strictly less (resp., weakly larger) than u_1.

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- $u' = s_{i_1} s_{i_2} \dots s_{i_a}$, and
- $u'' = u''_1 u''_2 \dots u''_b$ be the word over the alphabet $\{1, 2, \dots, q - c + 1\}$ order isomorphic with $s_{j_1} s_{j_2} \dots s_{j_b}$.

(2) Let $v' = \mathrm{rc}(u')$ and $v'' = \mathrm{rc}(u'')$.

(3) Let define $v \in [q]^n$ as : $v_1 = u_1$, and for each $1 \leq k \leq n-1$, v_{n-k+1} is defined as

- if k is the ℓth position in s (from left to right) with $s_k < u_1$, then $v_{n-k+1} = v'_{a-\ell+1}$,
- if k is the ℓth position in s (from left to right) with $s_k \geq u_1$, then $v_{n-k+1} = v''_{b-\ell+1} + c - 1$.

And finally, $p(u) = v$.

(3) Let define $v \in [q]^n$ as : $v_1 = u_1$, and for each $1 \leq k \leq n-1$, v_{n-k+1} is defined as

- if k is the ℓth position in s (from left to right) with $s_k < u_1$, then $v_{n-k+1} = v'_{a-\ell+1}$,
- if k is the ℓth position in s (from left to right) with $s_k \geq u_1$, then $v_{n-k+1} = v''_{b-\ell+1} + c - 1$.

And finally, $p(u) = v$.

An involution m on *the rearrangement of the letters of a word* : $(\mathrm{f}, \mathrm{des}, \mathrm{maj}, \mathrm{stat}, \mathrm{Ides})\, u = (\mathrm{f}, \mathrm{des}, \mathrm{stat}, \mathrm{maj}, \mathrm{Ides})\, m(u)$

m is the mapping obtained by replacing the point (2) in the previous definition for p by

(2') Let $v' = \phi(u')$ and $v'' = \phi(u'')$.

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Theorem

If $u \in [q]^n$, then

$$(\mathrm{f}, \mathrm{des}, \mathrm{stat}, \mathrm{maj}, \mathrm{Ides})\, u = (\mathrm{f}, \mathrm{des}, \mathrm{maj}, \mathrm{stat}, \mathrm{Ides})\, m(u).$$

Proof

By construction, $\mathrm{f}\, m(u) = \mathrm{f}\, u$.
Also, $\mathrm{Ides}\, \mathrm{flat}\, m(u) = \mathrm{Ides}\, \mathrm{flat}\, u$, and notations $\mathrm{Ides}\, v' = \mathrm{Ides}\, u'$ and $\mathrm{Ides}\, v'' = \mathrm{Ides}\, u''$, and so $\mathrm{Ides}\, m(u) = \mathrm{Ides}\, u$.
Furthemore, $\mathrm{Des}\, \mathrm{flat}\, m(u) = \mathrm{Des}\, \mathrm{flat}\, p(u)$, $\mathrm{Des}\, v' = \mathrm{Des}\, \mathrm{rc}(u')$ and $\mathrm{Des}\, v'' = \mathrm{Des}\, \mathrm{rc}(u'')$, and thus $\mathrm{Des}\, p(u) = \mathrm{Des}\, m(u)$. It follows that $\mathrm{des}\, m(u) = \mathrm{des}\, p(u) = \mathrm{des}\, u$, and
$\mathrm{comaj}\, m(u) = \mathrm{comaj}\, p(u)$ and thus
$\mathrm{stat}\, m(u) = \mathrm{stat}\, p(u) = \mathrm{maj}\, u$, and since m is an involution,
$\mathrm{maj}\, m(u) = \mathrm{stat}$.

Theorem

If $u \in [q]^n$, *then*

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Furthemore, $\mathrm{Des}\, \mathrm{flat}\, m(u) = \mathrm{Des}\, \mathrm{flat}\, p(u)$, $\mathrm{Des}\, v' = \mathrm{Des}\, \mathrm{rc}(u')$ and $\mathrm{Des}\, v'' = \mathrm{Des}\, \mathrm{rc}(u'')$, and thus $\mathrm{Des}\, p(u) = \mathrm{Des}\, m(u)$. It follows that $\mathrm{des}\, m(u) = \mathrm{des}\, p(u) = \mathrm{des}\, u$, and $\mathrm{comaj}\, m(u) = \mathrm{comaj}\, p(u)$ and thus $\mathrm{stat}\, m(u) = \mathrm{stat}\, p(u) = \mathrm{maj}\, u$, and since m is an involution, $\mathrm{maj}\, m(u) = \mathrm{stat}$.

Thank you for your attention !