(Un)ranking permutation classes

Nathanaël Hassler Vincent Vajnovszki
Laboratoire d'Informatique de Bourgogne
Université Bourgogne Europe
Dijon, France
nathanael.hassler@ens-rennes.fr vincent.vajnovszki@u-bourgogne.fr

Permutations avoiding a pattern of length three are enumerated by the Catalan numbers. In this work, we present methods for ranking and unranking such permutations in lexicographic or colexicographic order.

1 Introduction

For an ordered set of combinatorial objects we can ask which position in the ordered list of objects a given object has, or conversely determine the objet in a given position. Formally for a total ordered set C we have:

Definition 1.1. The *ranking* over C is the function

$$\text{rank}_C : C \to \{0, 1, \dots, |C| - 1\}$$

that maps x to the number of elements of C that are less than x. The *unranking* is the inverse function

$$\text{unrank}_C : \{0, 1, \dots, |C| - 1\} \to C$$

that maps p to the object x such that $\text{rank}_C(x) = p$.

Here, we restrict ourselves to sets C of restricted permutations of the same length in one-line representation, ordered lexicographically or colexicographically. Whenever the set C is clear from the context, the subscripts in the functions rank and unrank will be omitted.

Lehmer [8] may have been the first to suggest studying combinatorial objects by imposing an order on them and then developing corresponding ranking and unranking algorithms. Ruskey in his book [10] considers such algorithms for unrestricted permutations in both (co)lexicographic order and Steinhaus–Johnson–Trotter Gray code order. Among recent contributions in the field, we can cite [2, 3], and more particularly [5] for ranking and unranking algorithms for restricted permutations (derangements and ménage permutations).

Here, we are interested in ranking and unranking algorithms for simple yet non-trivial combinatorial classes, namely permutations avoiding a pattern of length three.

Aside from their intrinsic theoretical interest, such methods also have practical value. Indeed, given an unranking method, the ordered list can be computed as

$$\text{unrank}(0), \text{unrank}(1), \text{unrank}(2), \dots$$

and in the context of random sampling, $\text{unrank}(p)$ yields a random object as long as p is a random integer less than the cardinality of the combinatorial class. In addition, having a ranking method, the successor of a given object x can be computed as $\text{unrank}(\text{rank}(x) + 1)$. Even though these techniques are not efficient

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in general, they can provide hints for solutions to the underlying problems and enhance understanding of the concerned objects.

A permutation of the set $[n] = \{1,2,\dots,n\}$ is a bijection from $[n]$ onto itself, and $\mathfrak{S}_n$ denotes the set of such permutations. We will represent permutations by their one-line notation, that is, the permutation $\pi \in \mathfrak{S}_n$ by the word $\pi(1)\pi(2)\cdots\pi(n) \in [n]^n$.

Let k and n be integers, $0 < k \le n$, and $\sigma = \sigma(1)\sigma(2)\cdots\sigma(k) \in \mathfrak{S}_k$ and $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$ be two permutations. One says that π contains σ if π contains a (not necessarily contiguous) subsequence $\pi(i_1)\pi(i_2)\cdots\pi(i_k)$, $i_1 < i_2 < \cdots < i_k$, order isomorphic to σ; otherwise one says that π avoids σ, or π is σ-avoiding. In this context σ is called a (classical) *pattern*, and the set of length-n permutations avoiding the pattern σ is denoted by $\mathrm{Av}_n(\sigma)$, and $\mathrm{Av}(\sigma) = \cup_{n\ge 0}\mathrm{Av}_n(\sigma)$.

The lexicographic order on $[n]^n$ induces the lexicographic order on permutations in $\mathfrak{S}_n$ in their one-line representation, and it generalizes to words over any ordered alphabet. For two same length words $v = v_1v_2\dots v_n$ and $w = w_1w_2\dots w_n$ one says that v is less than w in *colexicographic order* if $v_nv_{n-1}\dots v_1$ is less than $w_nw_{n-1}\dots w_1$ in lexicographic order.

In the following we need the next easy to check result.

Remark 1.2. If C is a set of length-n words over an ordered alphabet, then the rank of $w = w_1w_2\dots w_n \in C$, in lexicographic order, is

$$\mathrm{rank}(w) = \sum_{i=1}^{n} |E_i|, \tag{1}$$

where for each i, E_i is the set of words $u_1u_2\dots u_n \in C$ with $u_j = w_j$ for $j < i$ and $u_i < w_i$.

The number of permutations in $\mathfrak{S}_n$ avoiding a pattern of length three is given by the Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$, see [7], which is sequence A000108 in [4]. Often it is convenient to represent pattern avoiding permutations $\pi \in \mathfrak{S}_n$ by an $n\times n$ array with a dot in each one of the squares $(i,\pi(i))$. See Figures 1 (b) and (c) for two examples.

2 Avoiding the 231 pattern

The *reduction* of a word α over a finite alphabet $A \subset \mathbb{N}$ is obtained by replacing each occurrence of the smallest symbol by 1, each occurrence of the second smallest symbol by 2, and so on. The *direct sum* of two permutations $\sigma \in \mathfrak{S}_k$ and $\tau \in \mathfrak{S}_\ell$, denoted by $\sigma \oplus \tau$, is the permutation $\pi \in \mathfrak{S}_{k+\ell}$ where $\pi(1)\pi(2)\cdots\pi(k) = \sigma$ and $\pi(k+1)\pi(k+2)\cdots\pi(k+\ell)$ is a permutation of $\{k+1,k+2,\dots,k+\ell\}$ that reduces to τ. Similarly, the *skew sum* of σ and τ, denoted by $\sigma \ominus \tau$, is the permutation $\pi \in \mathfrak{S}_{k+\ell}$ where $\pi(1)\pi(2)\cdots\pi(k)$ is a permutation of $\{\ell+1,\ell+2,\dots,\ell+k\}$ that reduces to σ and $\pi(k+1)\pi(k+2)\cdots\pi(k+\ell) = \tau$. With this notation, if $\pi = 1\ominus\sigma$, we have $\pi(1) = k+1$.

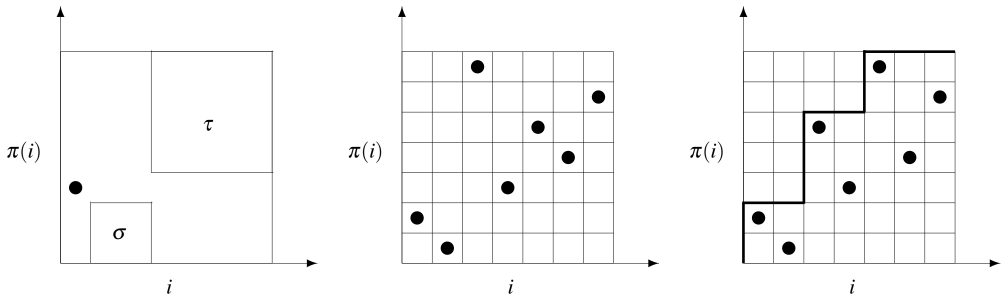
The next easy to understand fact is folklore.

Fact 2.1. A permutation avoids 231 if and only if it has the form $(1\ominus\sigma)\oplus\tau$, where σ and τ are (possibly empty) 231-avoiding permutations.

See Figure 1 (a) for the general shape of a 231-avoiding permutation, and (b) for the 231-avoiding permutation $2173546 = (1\ominus 1)\oplus(1\ominus(1\oplus((1\ominus 1)\oplus 1)))$.

Proposition 2.2. *Let $\pi = \pi(1)\pi(2)\cdots\pi(n) = (1\ominus\sigma)\oplus\tau$ be a length-n 231-avoiding permutation, with $\sigma \in \mathrm{Av}_k(231)$ and $\tau \in \mathrm{Av}_\ell(231)$ (and thus, $n = k+\ell+1$). Then the rank of π, in lexicographical order, is given recursively by*

$$\mathrm{rank}_{\mathrm{Av}_n(231)}(\pi) = \sum_{i=1}^{\pi(1)-1} c_i\cdot c_{n-i-1} + \mathrm{rank}_{\mathrm{Av}_k(231)}(\sigma)\cdot c_\ell + \mathrm{rank}_{\mathrm{Av}_\ell(231)}(\tau).$$



(a) (b) (c)

Figure 1: The arrays representation of: (a) the permutation $\pi = (1 \ominus \sigma) \oplus \tau$; (b) the 231-avoiding permutation 2173546; (c) the 321-avoiding permutation 2153746 corresponding to the Dyck word *bbaabbbaabbaaa* which codes the over-diagonal path in bold.

Proposition 2.3. *Let n, p be integers, $0 \le p < c_n$. The permutation* $\mathrm{unrank}_{\mathrm{Av}_n(231)}(p)$ *is obtained recursively as follows. Determine the smallest k such that*

$$p < \sum_{i=0}^{k} c_i c_{n-i-1}$$

and denote

$s := \sum_{i=0}^{k-1} c_i c_{n-i-1}$,

$u := \lfloor \frac{p-s}{c_{n-k}} \rfloor$, *and*

$v := p - s - u \cdot c_{n-k}$.

Then $\mathrm{unrank}_{\mathrm{Av}_n(231)}(p) = (1 \ominus \sigma) \oplus \tau$ *where*

$\sigma = \mathrm{unrank}_{\mathrm{Av}_{k-1}(231)}(u)$, *and*

$\tau = \mathrm{unrank}_{\mathrm{Av}_{n-k}(231)}(v)$.

In fact, the value k defined in the previous proposition is the first entry of $\mathrm{unrank}_{\mathrm{Av}_n(231)}(p)$.

3 Avoiding the 321 pattern

A *left-to-right maximum position* (or *lrmp* for short) in a permutation π is an index i such that, for all $j < i$, we have $\pi(j) < \pi(i)$. The value $\pi(i)$ is then called a *left-to-right maximum value* (or *lrmv* for short). An *excedance* is an index i such that $\pi(i) > i$. It is known (see, e.g., [9]) that a permutation is 321-avoiding if and only if both the subsequence of its left-to-right maximum values and the subsequence of its remaining elements are increasing. Moreover, such a permutation is uniquely determined by the set $\{(i, \pi(i))\}$, where i ranges over its left-to-right maximum positions. Even though we will not use it later on, it is worth mentioning that, for a 321-avoiding permutation, the left-to-right maxima positions coincide with the excedances of the permutation.

Another combinatorial class closely related to length-3 pattern-avoiding permutations is that of *Dyck words*. Let D_n denote the set of Dyck words of semilength n, that is, words $w \in \{a,b\}^{2n}$ containing the same number of occurrences of a and b, and such that every prefix of w contains at least as many b's as a's. (We adopt this unusual notation for convenience; traditionally, the roles of a and b are interchanged.) Dyck words in D_n are counted by the Catalan number c_n.

The bijection $\psi : \mathrm{Av}_n(321) \to D_n$ appeared originally in [7, p. 89] in a slightly different form. Consider the array representation corresponding to $\pi \in \mathrm{Av}_n(321)$ and the path with *east* and *north* steps along the edges of the array that goes from the lower-left corner to the upper-right corner of the array, leaving all the dots to the right and remaining always as close to the main diagonal as possible. Let P be the resulting path. Then $\psi(\pi)$ can be obtained from P just by reading its steps, a north step corresponding to the letter b and an east step to the letter a. See Figure 1 (c) for an example.

The next proposition says that ψ preserves the lexicographic order.

Proposition 3.1. *If $\pi, \tau \in \mathrm{Av}_n(321)$ with $\pi < \tau$ in lexicographic order, then $\psi(\pi) < \psi(\tau)$ in lexicographic order as well.*

Ballot sequences are prefixes of Dyck words. We denote by $B(i,j)$ the set of ballot sequences with i occurrences of b and j occurrences of a. Similarly, we denote by $\bar{B}(i,j)$ the set of *suffixes of Dyck words*.

Clearly, $B(i,j)$ is empty if $i < j$, and there is a unique word in $B(i,0)$, namely b^i. Moreover, if we ask to obtain a ballot sequence $w \in B(i,j)$ by appending a single letter to a shorter one, then

- if $i = j$, this can be obtained in a unique way: $w = ua$ for some $u \in B(i,j-1)$, and
- if $0 < j < i$, this can be obtained in two different ways, namely $w = ua$, for some $u \in B(i,j-1)$, and $w = ub$, for some $u \in B(i-1,j)$.

The considerations above (see for instance [6] and the references therein) lead us to the following recursive definition of $t(i,j)$, the cardinality of $B(i,j)$:

$$t(i,j) = \begin{cases} 0 & \text{if} \quad j > i, \\ 1 & \text{if} \quad j = 0, \\ t(i,j-1) & \text{if} \quad j = i > 0, \\ t(i-1,j) + t(i,j-1) & \text{if} \quad 0 < j < i. \end{cases} \tag{2}$$

The sequence $\big(t(i,j)\big)_{0 \le j \le i}$ is the sequence A009766 in [4], which has the closed form:

$$t(i,j) = \binom{i+j}{i} \frac{i-j+1}{i+1}, \text{ for } 0 \le j \le i, \tag{3}$$

and its first values are shown in Table 1.

Proposition 3.2 ([1])**.** *For the double indexed sequence $s(i,j) = \sum_{k=0}^{j} t(i,k)$ we have*

$$s(i,j) = t(i+1,j).$$

Representing a 321-avoiding permutation by both the sequence of its left-to-right maxima and the corresponding Dyck word (see Figure 2), we have:

Proposition 3.3. *Let π be a length-n 321-avoiding permutation, and let $(m_i, \ell_i)_{i=1}^k$ be the sequence where the m_i's are the* lrmp*'s of π, and $\ell_i = \pi(m_i)$ are the* lrmv*'s. Then,*

$$\mathrm{rank}_{\mathrm{Av}_{321}}(\pi) = \sum_{i=1}^{k} \left(s(n-m_i+1, n-\max\{\ell_{i-1}, m_i\}) - s(n-m_i+1, n-\ell_i+1) \right),$$

where for convenience $\ell_0 = 0$.

$i \setminus j$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2	2					
3	1	3	5	5				
4	1	4	9	14	14			
5	1	5	14	28	42	42		
6	1	6	20	48	90	132	132	
7	1	7	27	75	165	297	429	429

Table 1: The first values of the double indexed sequence $\big(t(i,j)\big)_{0\le j\le i}$.

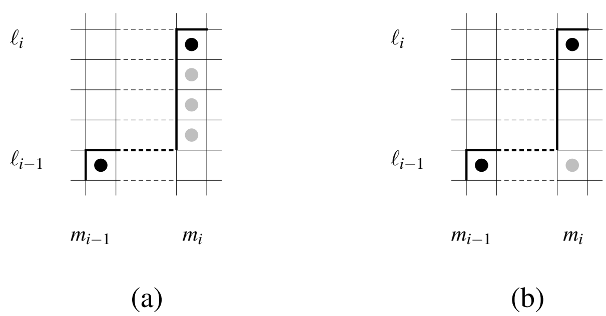


Figure 2: The black dots correspond to two consecutive *lrmp*'s, m_{i-1} and m_i, of a 321-avoiding permutation π, together with their *lrmv*'s $\ell_{i-1} = \pi(m_{i-1})$ and $\ell_i = \pi(m_i)$. Let σ be another 321-avoiding permutation of the same length that is smaller than π in lexicographic order and has the same first $(i-1)$ *lrmp*'s $m_1, m_2, \ldots, m_{i-1}$, and the same first $(i-1)$ *lrmv*'s, that is $\sigma(m_j) = \pi(m_j)$ for $j < i$. (a) The gray dots indicate the possible values for the ith *lrmv* of σ if m_i is still an *lrmp* of σ, with $\sigma(m_i) < \ell_i$. (b) The gray dot indicates the value of $\sigma(m_i)$ if m_i is no longer an *lrmp* of σ (and thus $\ell_{i-1} \ge m_i$).

Algorithm 1 Procedure for computing the Dyck word of semilength n and rank p. Here $\bar{t}(i,j) = \mathrm{card}(\overline{B}(i,j))$, so $\bar{t}(i,j) = t(j,i)$.

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procedure UNRANKDYCK(n, p: integer)
  global d: array
  $i \leftarrow n$, $j \leftarrow n$
  for k **from** 1 **to** $2n$ **do**
    if $i = j$ **or** $p \ge \bar{t}(i-1,j)$ **then**
      $d_k \leftarrow$ b; $p \leftarrow p - \bar{t}(i-1,j)$; $i \leftarrow i-1$
    else $d_k \leftarrow$ a; $j \leftarrow j-1$
    end if
end procedure

Let π be a 321-avoiding permutation of length n, and let $w = w_1w_2\dots w_{2n}$ be its corresponding Dyck word, that is $w = \Psi(\pi) \in D_n$. Obviously, the number of left-to-right maxima in π is equal to the number of occurrences of the factor ba (i.e., number of peaks) in w. Moreover, if $w_{j-1}w_j$ is the ith occurrence of the factor ba, then the ith left-to-right maximum (m_i, ℓ_i) of π is given by $m_i = |w_1w_2\dots w_j|_a$ and $\ell_i = |w_1w_2\dots w_j|_b$.

Proposition 3.4. *Let n, p be integers with $0 \le p < c_n$. The permutation* $\mathrm{unrank}_{\mathrm{Av}_n(321)}(p)$ *is obtained as follows:*

- *Run the* UNRANKDYCK *procedure in Algorithm 1 to construct the Dyck word w of semilength n and rank p.*
- *Using the considerations above, construct the set $\{(m_i, \pi(m_i))\}_{i=1}^{k}$ of left-to-right maxima (values, positions) of the permutation $\Psi^{-1}(w)$.*
- *Finally, construct the permutation* $\mathrm{unrank}_{\mathrm{Av}_n(321)}(p)$.

4 Trivial symmetries

Here, based on the previous results and the trivial symmetries between permutation classes, we derive ranking and unranking algorithms for the remaining length-3 avoiding patterns: 123, 132, 213, and 312.

Let $\mathfrak{c}$ be the complement operation on $\mathfrak{S}_n$, that is the involution defined as: $\sigma = \mathfrak{c}(\pi)$ if $\sigma(i) = n+1-\pi(i)$ for $1 \le i \le n$. Clearly, the complement operation $\mathfrak{c}$ induces a bijection between $\mathrm{Av}_n(231)$ and $\mathrm{Av}_n(213)$, and between $\mathrm{Av}_n(321)$ and $\mathrm{Av}_n(123)$. Moreover, if $\pi < \rho$ in lexicographic order, then $\mathfrak{c}(\rho) < \mathfrak{c}(\pi)$ in lexicographic order. With the considerations above we have:

Proposition 4.1. *Let τ be one of the length-3 patterns 213 or 123 (and so, $\mathfrak{c}(\tau)$ is 231 or 321). Then, for $n \ge 0$ and*

- *for $\pi \in \mathrm{Av}_n(\tau)$, $\mathrm{rank}_{\mathrm{Av}_n(\tau)}(\pi) = c_n - \mathrm{rank}_{\mathrm{Av}_n(\mathfrak{c}(\tau))}(\mathfrak{c}(\pi)) - 1$, and*
- *for an integer p, $0 \le p < c_n$, $\mathrm{unrank}_{\mathrm{Av}_n(\tau)}(p) = \mathrm{unrank}_{\mathrm{Av}_n(\mathfrak{c}(\tau))}(c_n - p - 1)$.*

Let now $\mathfrak{r}$ be the reverse operation on $\mathfrak{S}_n$, that is the involution on $\mathfrak{S}_n$ defined as: $\sigma = \mathfrak{r}(\pi)$ if $\sigma(i) = \pi(n+1-i)$ for $1 \le i \le n$. The reverse operation $\mathfrak{r}$ induces a bijection between $\mathrm{Av}_n(231)$ and $\mathrm{Av}_n(132)$, and between $\mathrm{Av}_n(213)$ and $\mathrm{Av}_n(312)$. Moreover, if $\pi < \rho$ in lexicographic order, then $\mathfrak{r}(\pi) < \mathfrak{r}(\rho)$ in colexicographic order. Denoting rank^* (resp. unrank^*) the ranking (resp. unranking) function in colexicographic order we have:

Proposition 4.2. *Let τ be one of the length-3 patterns 132 or 312 (and so, $\mathfrak{r}(\tau)$ is 231 or 213). Then, for $n \ge 0$ and*

- *for $\pi \in \mathrm{Av}_n(\tau)$, $\mathrm{rank}^*_{\mathrm{Av}(\tau)}(\pi) = \mathrm{rank}_{\mathrm{Av}(\mathfrak{r}(\tau))}(\mathfrak{r}(\pi))$,*
- *for an integer p, $0 \le p < c_n$, $\mathrm{unrank}^*_{\mathrm{Av}(\tau)}(p) = \mathfrak{r}(\mathrm{unrank}_{\mathrm{Av}(\mathfrak{r}(\tau))}(p))$.*

Open problems and future directions:

Even when the counting sequences for permutations avoiding multiple patterns of length three are trivial (e.g., 2^{n-1}, Fibonacci numbers), ranking and unranking methods in lexicographic order still seem less straightforward. It would also be interesting to consider the avoidance of patterns of size greater than three. Finally, the efficient implementation of these methods remains an open problem.

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