A new Euler-Mahonian constructive bijection

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- 1968: using a constructive bijection by Foata

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Main results

- We give new constructive bijection (ϕ) between *permutations* with a given
  - number of inversions, and
  - major index
- Introduce a new statistic, mix, related to the Lehmer code
- Show that the bistatistic (mix, INV) is Euler-Mahonian
- Introduce the *McMahon code* for permutations which is the major-index counterpart of Lehmer code and show how the two codes are related

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- A statistic on $\mathfrak{S}_n$ is a function

$$\mathfrak{S}_n \to \mathbb{N}$$

- bistatistic is a function

$$\mathfrak{S}_n \to \mathbb{N} \times \mathbb{N}$$

- (i, j) is an inversion of $\pi \in \mathfrak{S}_n$ if $i < j$ but $\pi_i > \pi_j$

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- $\operatorname{INV} \pi = \operatorname{card}\{(i,j) \mid 1 \leq i < j \leq n, \pi_i > \pi_j\}$.

Lehmer's code

Definition

An integer sequence $t_1 t_2 \ldots t_n$ is subexcedent if

$$0 \leq t_i \leq i - 1$$

The set of n-length subexcedent sequences is

$$S_n = \{0\} \times \{0, 1\} \times \ldots \times \{0, 1, \ldots, n - 1\}.$$

Definition

The Lehmer code is a bijection

$$code : \mathfrak{S}_n \rightarrow S_n$$

which maps

$$\pi_1\pi_2\ldots\pi_n \mapsto t_1t_2\ldots t_n$$

where t_i is the number of inversions (j, i) in π

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Example

$$\pi \quad = \quad 6 \quad 5 \quad 2 \quad 4 \quad 1 \quad 3$$

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$$\begin{array}{rcccccccc} & & \pi & = & 6 & 5 & 2 & 4 & 1 & 3 \\ code(\pi) & = & t & = & 0 & 1 & 2 & 2 & 4 & 3 \end{array}$$

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Example

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$$\text{INV}\,\pi = \sum_{i=1}^{n} t_i$$

For n, k and u

$$0 \leq k < u \leq n$$

let define $\rho_{u,k} \in \mathfrak{S}_n$ as the permutation obtained from the identity in $\mathfrak{S}_n$ after a left circular shift of the segment of length $k+1$ ending at position u

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Example (in $\mathfrak{S}_5$)

$$\rho_{3,1} = 1\,3\,2\,4\,5$$

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Example (in $\mathfrak{S}_5$)

$$\rho_{3,1} = 1\,3\,2\,4\,5$$

$$\rho_{3,2} = 2\,3\,1\,4\,5$$

For n, k and u

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Example (in $\mathfrak{S}_5$)

$$\rho_{3,1} = 1\,3\,2\,4\,5$$

$$\rho_{3,2} = 2\,3\,1\,4\,5$$

$$\rho_{5,3} = 1\,3\,4\,5\,2$$

$$\rho_{u,k} =$$

$$\begin{pmatrix} 1 & \cdots & u-k-1 & u-k & u-k+1 & \cdots & u & u+1 & \cdots & n \\ 1 & \cdots & u-k-1 & u-k+1 & u-k+2 & \cdots & u-k & u+1 & \cdots & n \end{pmatrix}$$

Remark

Every permutation $\pi \in \mathfrak{S}_n$ *can be recovered from its Lehmer code* $t = t_1 t_2 \ldots t_n \in S_n$ *by*

$$\begin{aligned} \pi &= \rho_{n,t_n} \cdot \rho_{n-1,t_{n-1}} \cdot \ldots \cdot \rho_{i,t_i} \cdot \ldots \cdot \rho_{2,t_2} \cdot \rho_{1,t_1} \\ &= \prod_{i=n}^{1} \rho_{i,t_i} \end{aligned}$$

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$$code^{-1}(t_1 t_2 \ldots t_n) = \prod_{i=n}^{1} \rho_{i,t_i}$$

target: 6 5 2 4 1 3

1 2 3 4 5 6

$code(6\,5\,2\,4\,1\,3) =$

target: 6 5 2 4 1 3

1 2 3 4 5 6

$code(6\,5\,2\,4\,1\,3) =$

target: 6 5 2 4 1 3

1	2	3	4	5	6
1	2	4	5	6	3

$code(6\,5\,2\,4\,1\,3) =$

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$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 4 & 5 & 6 & 3 \end{matrix} = \rho_{6,3}$$

$code(6\,5\,2\,4\,1\,3) = 3$

target: 6 5 2 4 1 3

1	2	3	4	5	6
1	2	4	5	6	3

$= \rho_{6,3}$

$code(6\,5\,2\,4\,1\,3) = 3$

target: 6 5 2 4 1 3

1	2	3	4	5	6
1	2	4	5	6	3
2	4	5	6	1	3

$= \rho_{6,3}$

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target: 6 5 2 4 1 3

1	2	3	4	5	6
1	2	4	5	6	3
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$= \rho_{6,3}$

$= \rho_{6,3} \cdot \rho_{5,4}$

$code(6\,5\,2\,4\,1\,3) = 4\,3$

target: 6 5 2 4 1 3

1	2	3	4	5	6
1	2	4	5	6	3
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$= \rho_{6,3}$

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$= \rho_{6,3} \cdot \rho_{5,4}$

$= \rho_{6,3} \cdot \rho_{5,4} \cdot \rho_{4,2}$

$code(6\,5\,2\,4\,1\,3) = 2\,4\,3$

target: 6 5 2 4 1 3

1	2	3	4	5	6
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2	5	6	4	1	3	$= \rho_{6,3} \cdot \rho_{5,4} \cdot \rho_{4,2}$
5	6	2	4	1	3	$= \rho_{6,3} \cdot \rho_{5,4} \cdot \rho_{4,2} \cdot \rho_{3,2}$

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5	6	2	4	1	3	$= \rho_{6,3} \cdot \rho_{5,4} \cdot \rho_{4,2} \cdot \rho_{3,2}$
6	5	2	4	1	3	$= \rho_{6,3} \cdot \rho_{5,4} \cdot \rho_{4,2} \cdot \rho_{3,2} \cdot \rho_{2,1}$

$code(6\,5\,2\,4\,1\,3) = 1\,2\,2\,4\,3$

target: 6 5 2 4 1 3

1	2	3	4	5	6	
1	2	4	5	6	3	$= \rho_{6,3}$
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$code(6\,5\,2\,4\,1\,3) = 0\,1\,2\,2\,4\,3$

For n, k and u

$$0 \leq k < u \leq n$$

let define $[[u,k]] \in \mathfrak{S}_n$ as the permutation obtained after k right circular shifts of the u-length prefix of the identity in $\mathfrak{S}_n$

Bijections ϕ and ψ

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Example (in $\mathfrak{S}_5$)

$$[\![3, 1]\!] = \underline{3\,1\,2}\,4\,5$$
$$[\![3, 2]\!] = \underline{2\,3\,1}\,4\,5$$
$$[\![5, 3]\!] = \underline{3\,4\,5\,1\,2}$$

$$[[u,k]] =$$

$$\begin{pmatrix} 1 & 2 & \cdots & k & k+1 & \cdots & u & u+1 & \cdots & n \\ u-k+1 & u-k+2 & \cdots & u & 1 & \cdots & u-k & u+1 & \cdots & n \end{pmatrix}$$

Definition

$$\psi : S_n \to \mathfrak{S}_n$$

$$\psi(t_1 t_2 \ldots t_n) =$$

$$[[n, t_n]] \cdot [[n-1, t_{n-1}]] \cdot \ldots \cdot [[i, t_i]] \cdot \ldots \cdot [[2, t_2]] \cdot [[1, t_1]] =$$

$$\prod_{i=n}^{1} [[i, t_i]]$$

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$$\prod_{i=n}^{1} [[i, t_i]]$$

- Every permutation in $\mathfrak{S}_n$ can be uniquely written as

$$\prod_{i=n}^{1} [[i, t_i]]$$

for some t_i's (next Lemma)

- $\{\rho_{i,k}\}_{0 \leq k < i \leq n}$ and $\{[[i,k]]\}_{0 \leq k < i \leq n}$ are both generating sets for $\mathfrak{S}_n$
- $[[i,k]]$ can be viewed as a MAJ *counterpart* of $\rho_{i,k}$

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Lemma

The function

$$\psi : S_n \rightarrow \mathfrak{S}_n$$

$$\psi(t_1 t_2 \ldots t_n) = \prod_{i=n}^{1} [[i, t_i]]$$

is a bijection

target: 5 2 1 6 4 3

1 2 3 4 5 6

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1	2	3	4	5	6
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1	2	3	4	5	6	
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$$= [[6,3]]$$
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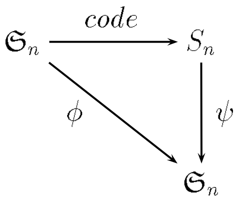


Definition

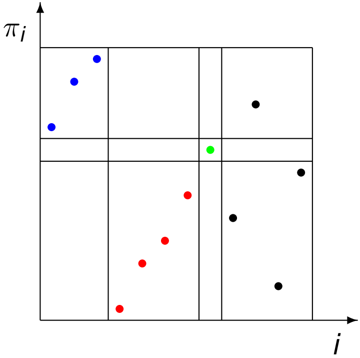
We say that $\pi \in \mathfrak{S}_n$ is k-separate, $1 \leq k \leq n$, if there exists an ℓ such that π can be written as the concatenation of three 'segments' (the first two of them possibly empty)

$$\pi = \pi_1 \pi_2 \dots \pi_\ell \pi_{\ell+1} \pi_{\ell+2} \dots \pi_{k-1} \pi_k \pi_{k+1} \dots \pi_n \tag{1}$$

with

- $\pi_i < \pi_j$ for all i and j, $1 \leq i < j \leq \ell$ or $\ell + 1 \leq i < j \leq k - 1$, and
- $\pi_i > \pi_k > \pi_j$ for all i and j, $1 \leq i \leq \ell < j \leq k - 1$.

By convention we consider that the identity in $\mathfrak{S}_n$ is $(n + 1)$-separate.



The permutation 9 11 12 1 3 4 6 8 5 10 2 7 $\in \mathfrak{S}_{12}$ is 8-separable (and so j-separable for $1 \leq j < 8$).

Lemma

Let $\pi \in \mathfrak{S}_n$ be k-separate
For i, v with $0 < v < i < k$

$$\sigma = \pi \cdot [[i, v]]$$

a) *If π has no descents at the left of k, then v is the unique descent in σ at the left of k*

Otherwise, let ℓ be the (unique) descent in π at the left of k. In this case:

b) *if $v \leq i - \ell$, then $\ell + v$ is the unique descent in σ at the left of k*

c) *if $v > i - \ell$, then σ has two descents at the left of k: i and $v - i + \ell$*

Corollary

If π *is* k*-separate, then for all* i *and* v, $0 \le v < i < k$

- $\pi \cdot [[i, v]]$ *is* i*-separate*
- $\mathrm{MAJ}\,(\pi \cdot [[i, v]]) = \mathrm{MAJ}\,\pi + v$

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Theorem

For every $\pi \in \mathfrak{S}_n$ *we have* $\text{MAJ}\,\phi(\pi) = \text{INV}\,\pi$.

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Corollary

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\text{MAJ}\,\sigma} = \sum_{\sigma \in \mathfrak{S}_n} q^{\text{INV}\,\sigma}$$

The mix statistic

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\mathsf{INV}\,\sigma} = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathsf{MAJ}\,\sigma}$$

The mix statistic

$$\sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{mix}\,\sigma}\; q^{\mathrm{INV}\,\sigma} = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}\,\sigma}\; q^{\mathrm{MAJ}\,\sigma}$$

Definition

For $t = t_1 t_2 \ldots t_n \in S_n$ let $b = b_1 b_2 \ldots b_{n-1}$ be a binary sequence with

- $$\sum_{j=1}^{n-1} j \cdot b_j = \sum_{j=1}^{n} t_j$$

b is called the multi-radix (binary) array of t

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- for all $i \geq 1$
$$\sum_{j=i}^{n} t_j - i < \sum_{j=i}^{n-1} j \cdot b_j \leq \sum_{j=i}^{n} t_j,$$

b is called the multi-radix (binary) array of t

Example

t	multi-radix sequence of t
0 0 0 1 4	1 0 0 1
0 0 2 0 3	0 1 1 0
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```
r := 0;
for i := n downto 1 do
      r := r + t_i;
      if i ≤ r;
      then b_i := 1; r := r − i;
      else b_i := 0;
      endif
enddo
```

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- For $\pi \in \mathfrak{S}_n$, $\text{mix}\,\pi = \text{mix}\,t$ where t is the Lehmer code of π

  Formally, $\text{mix}\,\pi = \text{mix}\,\mathit{code}(\pi)$

and we extend des and mix statistics to set-valued functions.

Let

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$$\mathrm{des}\,\pi = \mathrm{card}\,\mathrm{D}(\pi),$$

$$\mathrm{MAJ}\,\pi = \sum_{i \in \mathrm{D}(\pi)} i$$

Define

$$\mathrm{M} : \mathfrak{S}_n \rightarrow 2^{\{1,2,\ldots,n-1\}}$$

as the set of positions of 1-bits in the multi-radix array of the Lehmer code of π

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$$\operatorname{mix} \pi = \operatorname{card} \mathrm{M}(\pi),$$

$$\mathrm{INV}\, \pi = \sum_{i \in \mathrm{M}(\pi)} i$$

Theorem

For every subset T of $\{1, 2, \cdots, n-1\}$, we have

$$\mathrm{card}\{\pi \in \mathfrak{S}_n \,|\, \mathsf{M}(\pi) = T\} = \mathrm{card}\{\tau \in \mathfrak{S}_n \,|\, \mathsf{D}(\tau) = T\}.$$

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$$\mathsf{M}(\pi) = \mathsf{D}(\phi(\pi))$$

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The bistatistic (mix, INV) *is Euler-Mahonian, or equivalently,*

$$\sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{mix}\,\sigma} q^{\mathrm{INV}\,\sigma} = \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{des}\,\sigma} q^{\mathrm{MAJ}\,\sigma}.$$

- mix can be seen just as a new Eulerian partner for inversions [M. SKANDERA,(2001)]
- Another Euler-Mahonian bistatistic is (exc, den) with
  - exc = the excedance number
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  thus den is a Mahonian partner for the excedance number [D. FOATA AND D. ZEILBERGER (1990)]

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From Lehmer code to McMahon code

Definition

For $\pi \in \mathfrak{S}_n$ the sequence $s = s_1 s_2 \ldots s_n$ such that

$$\pi = \prod_{i=n}^{1} [[i, s_i]]$$

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how the Lehmer code and McMahon code are related?

Definition

Let

$$\Delta : S_n \rightarrow S_n$$

$$t_1 t_2 \ldots t_n \mapsto \Delta(t) = s_1 s_2 \ldots s_n$$

- $s_n = t_n$, and
- $s_i = (t_{i+1} - t_i) \mod i$, for $1 \leq i \leq n-1$.

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Example

$\Delta(0\,1\,2\,2\,4\,3) = 0\,1\,0\,2\,4\,3$

$$\Delta^{-1} : S_n \to S_n$$

$$s_1 s_2 \ldots s_n \mapsto \Delta^{-1}(s) = t_1 t_2 \ldots t_n$$

- $t_n = s_n$, and
- $t_i = (t_{i+1} + s_i) \mod i$, for $1 \leq i \leq n - 1$.

Theorem

If $\pi \in \mathfrak{S}_n$ *has its Lehmer code* $t = t_1 t_2 \dots t_n$*, then its McMahon code is* $\Delta(t)$

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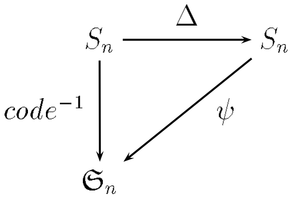

Corollary

For any $\pi \in \mathfrak{S}_n$*, the McMahon code* $s = s_1 s_2 \dots s_n$ *of* π *satisfies:*

- $s_i = \text{card}\,\{j \mid 1 \le j < i, \pi_j \in [\pi_{i+1}, \pi_i]\}$ *if* $\pi_{i+1} < \pi_i$*,*
- $s_i = \text{card}\,\{j \mid 1 \le j < i, \pi_j \notin [\pi_i, \pi_{i+1}]\}$ *elsewhere,*

with the convention that $\pi_{n+1} = n + 1$*.*

For a subexcedent sequence

$$t = t_1 t_2 t_3 \ldots t_{n-1} t_n \in S_n$$

define its complement t^c as the subexcedent sequence

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For a permutation

$$\pi = \pi_1 \pi_2 \pi_3 \dots \pi_{n-1} \pi_n$$

define its complement as

$$\pi^c = (n+1-\pi_1)(n+1-\pi_2)(n+1-\pi_3) \dots (n+1-\pi_{n-1})(n+1-\pi_n)$$

- $code^{-1}(t^c) = (code^{-1}(t))^c$

Lemma

For any $t \in S_n$, we have

(i) $\Delta(t^c) = (\Delta(t))^c$,

(ii) $\Delta^{-1}(t^c) = (\Delta^{-1}(t))^c$,

(iii) $\psi(t^c) = (\psi(t))^c$.

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Corollary

- $\mathrm{D}(\psi(t^c)) = \{1, 2, \ldots, n-1\} \setminus \mathrm{D}(\psi(t))$,
- $\mathrm{M}(code^{-1}(t)) = \mathrm{D}((code^{-1} \circ \Delta^{-1})(t))$.

Question: can the previous results be naturally generalized to multiset permutations?

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