The equidistribution of some vincular patterns on 132-avoiding permutations

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On $S_n(132)$, the vincular pattern based statistics

- $2\underline{31}$, $2\underline{13}$ and $\underline{21}3$,
- $\underline{23}1$, $3\underline{12}$,

have the same distribution.

On $S_n(132)$,

- $2\underline{31} \equiv 2\underline{13} \equiv \underline{21}3$,
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Up to trivial transformations, these patterns are the only length three proper (not classical nor adjacent) vincular patterns equidistributed on a set of permutations avoiding a classical length three pattern.

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Outline

- Notations and definitions
- Related works
- ϕ : $(2\underline{31}, 2\underline{13}, \text{rlmin}, \text{rlmax}) \equiv (2\underline{13}, 2\underline{31}, \text{rlmax}, \text{rlmin})$
- ψ: $(2\underline{31}, \text{des}) \equiv (2\underline{13}, \text{des})$
- μ : $(\underline{21}3, \text{des}, 12]) \equiv (2\underline{13}, \text{des}, 12])$
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Notations and definitions

A *permutation* of length n is a bijection from $\{1, 2, \ldots, n\}$ to itself; in *one-line notation*

$$\pi = \pi_1 \pi_2 \ldots \pi_n.$$

S_n = the set of permutations of length $n \geq 0$.
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Patterns

Let $\sigma \in S_k$, $\pi \in S_n$, $k \leq n$.

σ occurs as a (classical) pattern in π if there is a sequence

$$1 \leq i_1 < i_2 < \cdots < i_k \leq n$$

such that

$$\pi_{i_1} \pi_{i_2} \cdots \pi_{i_k}$$

is order-isomorphic with σ.

Example $\sigma = 231$ occurs in $\pi =$13452 and the three occurrences are

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Vincular patterns

Any pair of two adjacent letters in the pattern may now be underlined = corresponding letters in the permutation must be adjacent.

Example $\sigma = 2\underline{13}$ occurs in $\pi = 425163$ four times:

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multistatistic is a tuple of statistics.

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In a permutation π, π_i is a *right-to-left maximum* if

$$\pi_i > \pi_j \text{ for all } j > i$$

and the number of right-to-left maxima of π is denoted by rlmax π.

Similarly: *right-to-left minimum* and rlmin π.

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rlmax 416523

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st and st′ have the same distribution (are equidistributed) on S if, for any k,

$$\text{card}\{\pi \in S : \text{st}\,\pi = k\} = \text{card}\{\pi \in S : \text{st}'\,\pi = k\},$$

The multistatistics $(\text{st}_1, \text{st}_2, \ldots, \text{st}_p)$ and $(\text{st}'_1, \text{st}'_2, \ldots, \text{st}'_p)$ have the same distribution if, for any $k = (k_1, k_2, \ldots, k_p)$,

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For a permutation π and a patterns σ

$$(\sigma)\,\pi$$

denotes the number of occurrences of σ in π, and (σ) becomes a permutation statistic.

Example

- $(\underline{21})\,\pi = \text{des}\,\pi$
- $(21)\,\pi = \text{inv}\,\pi$

For a set of patterns $\{\sigma, \tau, \ldots\}$,

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Sum decomposition

- the *skew sum* of α and β, denoted $\alpha \ominus \beta$, is π of length $|\alpha| + |\beta|$:

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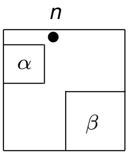

$\pi = (\alpha \oplus 1) \ominus \beta \quad \pi = \alpha \ominus (\beta \oplus 1)$

Fact (Folklore)

For a non-empty permutation π, the following are equivalent:

- π *avoids* 132,
- π *can uniquely be written as* $(\alpha \oplus 1) \ominus \beta$,
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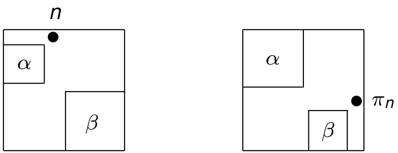
where α *and* β *are (possibly empty)* 132*-avoiding permutations.*



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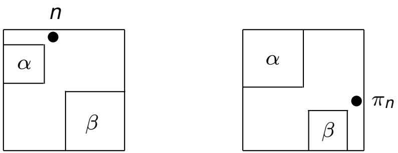

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Related works

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the equidistribution of some length three consecutive patterns involvement statistics on $S_n(312)$.

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2013: **Bóna**
(the surprising fact)

$$\sum_{\tau \in S_n(132)} (231)\,\tau = \sum_{\tau \in S_n(132)} (213)\,\tau$$

beside the pattern based statistics $(231) \neq (213)$ on $S_n(132)$.

$\textcolor{blue}{2012}$: **Homberger**

$$\sum_{\tau\in S_n(123)} (231)\,\tau = \sum_{\tau\in S_n(132)} (231)\,\tau$$

despite the pattern statistic (231) has different distribution on the two sets.

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2013: **Burnstein and Elizalde** gave (in a much more general context) the total number of occurrences of any vincular pattern of length three on $S_n(231)$.

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Bijection ϕ

$$\phi : S_n(132) \to S_n(132)$$

is recursively defined as:

- $\phi(\emptyset) = \emptyset$
- if
$$\pi = \alpha \ominus (\beta \oplus 1)$$
for some permutations $\alpha, \beta \in S(132)$, then
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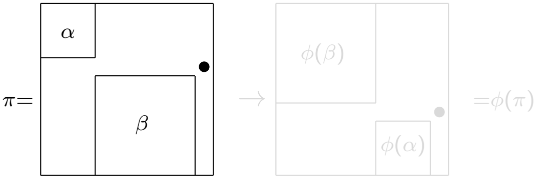

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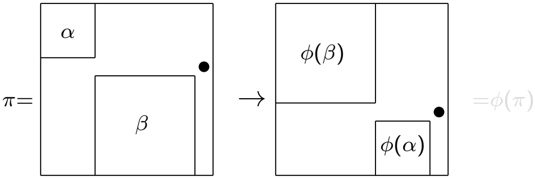

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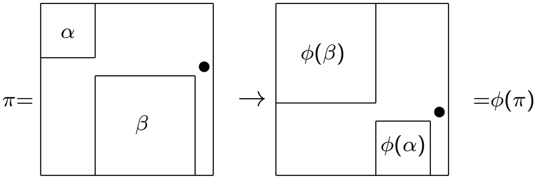

By induction we have:

Remark

- *If* $\pi \in S_n(132)$*, then* $\phi(\pi) \in S_n(132)$
- $\phi(\phi(\pi)) = \pi$.

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If $\pi \in S_n(132)$*, then*

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- $(2\underline{13})\,\phi(\pi) = (2\underline{31})\,\pi$,
- $\text{rlmin}\,\phi(\pi) = \text{rlmax}\,\pi$.

Theorem

If $\pi \in S_n(132)$*, then*

$$(2\underline{13}, 2\underline{31}, \text{rlmin}, \text{rlmax})\,\phi(\pi) = (2\underline{31}, 2\underline{13}, \text{rlmax}, \text{rlmin})\,\pi.$$

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- ψ: $(2\underline{31}, \mathrm{des}) \equiv (2\underline{13}, \mathrm{des})$
- μ: $(\underline{21}3, \mathrm{des}, 12]) \equiv (2\underline{13}, \mathrm{des}, 12])$
- $^{-1}$: $(\underline{23}1, 3\underline{12}, \mathrm{des}) \equiv (3\underline{12}, \underline{23}1, \mathrm{des})$

Bijection ψ

$$\psi : S_n(132) \to S_n(132)$$

- $\psi(\emptyset) = \emptyset$, $\psi(1) = 1$ and
- if $n \geq 2$, then $\psi(\pi)$ is defined according with three cases
  1. $\pi^{-1}(n) = 1$
  2. $1 < \pi^{-1}(n) < n$
  3. $\pi^{-1}(n) = n$

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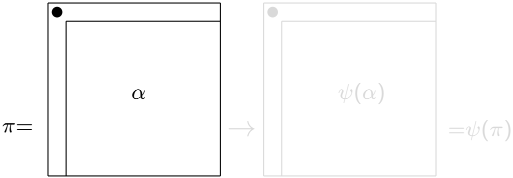

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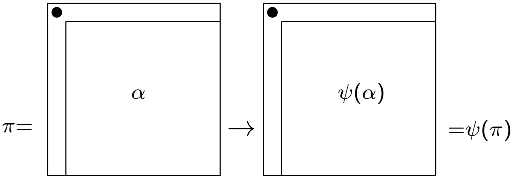

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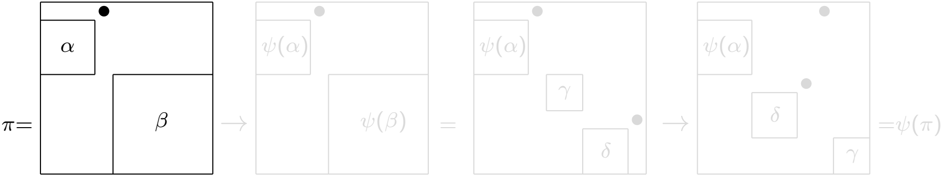

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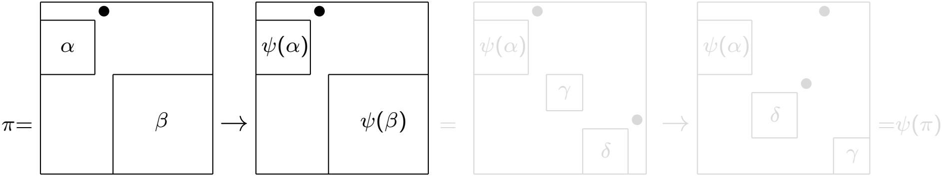

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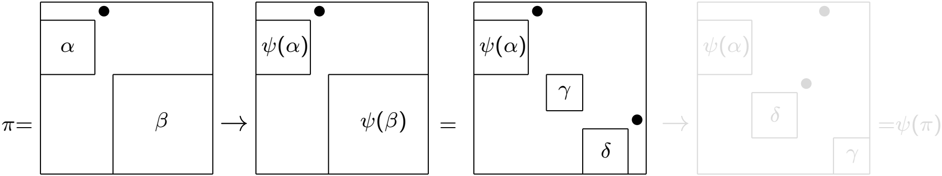

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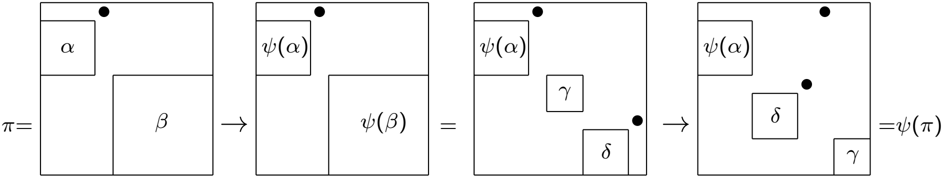

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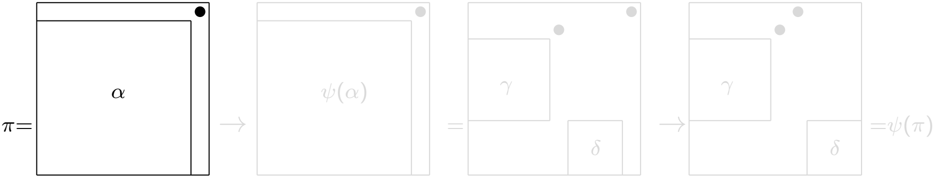

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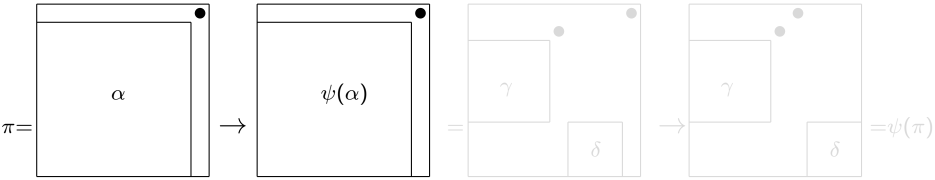

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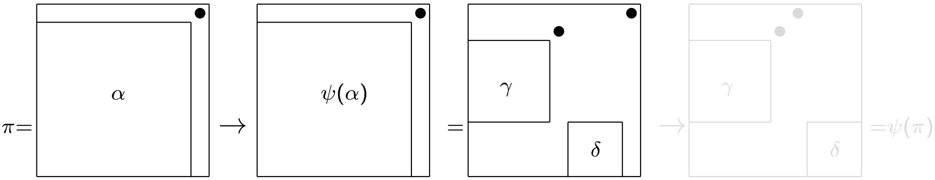

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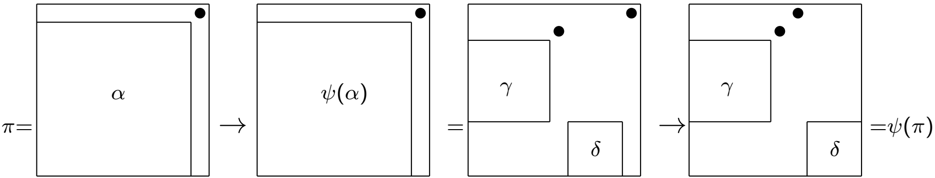

By induction on n:

- ψ *is well defined*
- ψ *is a bijection*
- $\text{des}\,\psi(\pi) = \text{des}\,\pi$

Theorem

$(2\underline{31}, \text{des})\,\psi(\pi) = (2\underline{13}, \text{des})\,\pi$

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- Notations and definitions
- Related works
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  $(2\underline{31}, 2\underline{13}, \text{rlmin}, \text{rlmax})$, and
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- Bijection μ: equidistribution of

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  $(\underline{23}1, 3\underline{12}, \text{des})$, and
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Bijection μ

$$\mu : S_n(132) \to S_n(132)$$

- $\mu(\emptyset) = \emptyset$
- if
$$\pi = \alpha \ominus (\beta \oplus 1)$$
for some $\alpha, \beta \in S(132)$, then
$$\mu(\pi) = \mu(\alpha) \ominus (\mu(\psi(\beta)) \oplus 1).$$

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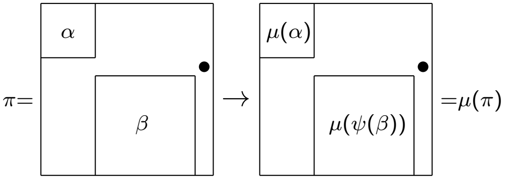

$$(2\underline{13} + 21])\,\pi = (2\underline{31} + \underline{21})\,\pi,$$

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Lemma

For any permutation π we have

$$(2\underline{13})\,\pi \oplus 1 = (2\underline{31} + \text{des})\,\pi.$$

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Theorem

For $\pi \in S_n(132)$

$$(\underline{21}3, \mathrm{des}, 12])\, \mu(\pi) = (2\underline{13}, \mathrm{des}, 12])\, \pi$$

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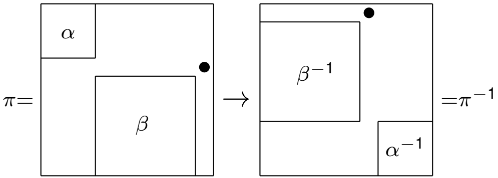

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Conclusions

On S_n,

- $2\underline{31} \equiv 2\underline{13} \neq \underline{21}3$,
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`http://fr.arxiv.org/pdf/1412.3512.pdf`

Grazie mólto!