Generalized Schröder permutations

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- $\mathcal{S}_n$ is the set of length $n$ permutations
- $\mathcal{S}_n(A)$ is the set of permutations in $\mathcal{S}_n$ avoiding each permutation in $A$
- $\mathcal{S}(A) = \bigcup_{n=0}^{\infty} \mathcal{S}_n(A)$
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*Permutations avoiding an increasing number of length-increasing forbidden subsequences*, 2000
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\[ \Gamma_m \subset S_m \]

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$\Gamma_m$ is the set of length $m$ permutations with fixed points in the last and last but one position
Example

\[ \Gamma_3 = \{123\} \]
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\[ \Gamma_4 = \{1234, 2134\} \]
Example

- $\Gamma_3 = \{123\}$ and so $\text{card}(S_n(\Gamma_3)) = c_n$, the $n$th Catalan number,
- $\Gamma_4 = \{1234, 2134\}$ and so $\text{card}(S_n(\Gamma_4)) = r_n$, the $n$th Schröder number
Barcucci et al. gave a multivariate generating function for the set of permutations in $\mathcal{S}(\Gamma_m)$ according with

- length
- left minima
- non-inversions
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In particular

Theorem (Barcucci et al.)

The generating function of the sequence $(\text{card}(\mathcal{S}_n(\Gamma_m)))_{n \geq 0}$
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- length
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In particular

**Theorem (Barcucci et al.)**

The generating function of the sequence $(\text{card}(\mathcal{S}_n(\Gamma_m)))_{n \geq 0}$ is

$$
\sum_{i=1}^{m-3} i! \cdot x^i + x^{m-4} \cdot (m-3)! \cdot \frac{1 - (m - 1)x - \sqrt{1 - 2 (m - 1)x + (m - 3)^2x^2}}{2}
$$
D. Kremer *Permutations with forbidden subsequences and a generalized Schröder number*, 2000
Postscript: “Permutations with forbidden subsequences and a generalized Schröder number” 2003
The context

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For $1 \leq s, t \leq m$, $s \neq t$, define $\Gamma_{m,s,t} \subset \mathfrak{S}_m$ by

$$\Gamma_{m,s,t} = \{ \sigma \in \mathfrak{S}_m | \sigma(s) = m - 1 \text{ and } \sigma(t) = m \}$$

In particular, $\Gamma_{m;m-1,m} = \Gamma_m$
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For $1 \leq s, t \leq m$, $s \neq t$, define $\Gamma_{m; s, t} \subset \mathfrak{S}_m$ by

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In particular, $\Gamma_{m; m-1, m} = \Gamma_m$

Theorem (Kremer 2000, 2003)

*For*

- $|t - s| \leq 2$, or
- $t \in \{1, m\}$

the cardinality of $\mathfrak{S}_n(\Gamma_{m; s, t})$ does not depend on $s$ and $t
The main result

We generalize these results by imposing that the second largest element of the length $m$ forbidden patterns occurs in *one of the last $p$ positions*.
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For $1 \leq j < m$ let define $\Sigma_{m,j} \subset S_m$ by:

$\sigma \in \Sigma_{m,j}$ iff

- $\sigma(1) = m$, and
- $\sigma(m + 1 - j) = m - 1$
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Example

$\Sigma_{4,1} = \{4123, 4213\}$
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- $\sigma(m + 1 - j) = m - 1$

**Example**

- $\Sigma_{4,1} = \{4123, 4213\}$
- $\Sigma_{4,2} = \{4132, 4231\}$
For $1 \leq p < m$ define $\Sigma_m^p \subset \mathcal{S}_m$ by

$$\Sigma_m^p = \bigcup_{j=1}^{p} \Sigma_{m,j}$$
Example

- \((m, p) = (2, 1)\)
  \[
  \Sigma^1_2 = \{21\} \text{ and } \text{card}(\mathcal{S}_n(\Sigma^1_2)) = 1
  \]
Example

- $(m, p) = (2, 1)$
  $\Sigma^1_2 = \{21\}$ and $\text{card}(\mathcal{G}_n(\Sigma^1_2)) = 1$

- $(m, p) = (3, 1)$
  $\Sigma^1_3 = \{312\}$ and $\text{card}(\mathcal{G}_n(\Sigma^1_3))$ is the Catalan number
Example

- \((m, p) = (2, 1)\)
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- \((m, p) = (3, 1)\)
  \[\Sigma_3^1 = \{312\} \text{ and } \text{card}(\mathcal{S}_n(\Sigma_3^1)) \text{ is the Catalan number}\]

- \((m, p) = (3, 2)\)
  \[\Sigma_3^2 = \{312, 321\} \text{ and } \text{card}(\mathcal{S}_n(\Sigma_3^2)) = 2^{n-1}\]
Example

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Example

- $(m, p) = (2, 1)$
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- $(m, p) = (4, 1)$
  $\Sigma^1_4 = \{4123, 4213\}$ and $\text{card}(\mathcal{G}_n(\Sigma^1_4))$ is the Schröder number

- $(m, p) = (4, 2)$
  $\Sigma^2_4 = \{4123, 4213, 4132, 4231\}$ and $\text{card}(\mathcal{G}_n(\Sigma^2_4))$ is the $(n - 1)$th central binomial coefficient $\binom{2n-2}{n-1}$
\[
\text{card}(\Sigma_m^p)
\]

<table>
<thead>
<tr>
<th>(m) (\backslash) (p)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<td></td>
<td></td>
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<tr>
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<td>(2^{n-1})</td>
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<td>4</td>
<td>Schröder</td>
<td>(\binom{2n-2}{n-1})</td>
<td>(2 \cdot 3^{n-2}) A025192</td>
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<tr>
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<td>A054872</td>
<td></td>
<td></td>
<td>(6 \cdot 4^{n-3}) A084509</td>
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</table>
We give a generating function for the set

\[ \mathcal{G}(\Sigma^p_m) \]
Generating trees
Production matrices
Generating trees: how $\mathcal{G}(\Sigma_m^p)$ can be recursively defined?

Production matrices
Generating trees: how $\mathcal{G}(\Sigma^p_m)$ can be recursively defined?

Production matrices: how this definition can be turned into a generating function?
A **succession** (or **ECO** rule) is a formal system consisting of a root $e_0$ (or axiom) and a set of *productions* of the form

$$(k) \rightsquigarrow (e_1(k))(e_2(k))\ldots(e_k(k))$$

Succession rule explains how an object of size $n$ can be uniquely expanded into several objects of size $n + 1$. 
The *sites* of $\pi \in \mathfrak{S}_n$ are the positions between two consecutive entries, before the first and after the last entry.
The *sites* of $\pi \in \mathcal{S}_n$ are the positions between two consecutive entries, before the first and after the last entry.

For a permutation $\pi \in \mathcal{S}_n(T)$, $i$ is an *active site* if the permutation obtained from $\pi$ by inserting $n + 1$ into its $i$th site is a permutation in $\mathcal{S}_{n+1}(T)$; we call such a permutation in $\mathcal{S}_{n+1}(T)$ a *son* of $\pi$. 

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Generalized Schröder permutations
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For any $\pi \in \mathcal{S}_n(T)$, by erasing $n$ in $\pi$ one obtains a permutation in $\mathcal{S}_{n-1}(T)$; or equivalently, any permutation in $\mathcal{S}_n(T)$ is obtained from a permutation in $\mathcal{S}_{n-1}(T)$ by inserting $n$ in one of its active sites.
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For a permutation $\pi \in \mathfrak{S}_n(T)$, $i$ is an **active site** if the permutation obtained from $\pi$ by inserting $n + 1$ into its $i$th site is a permutation in $\mathfrak{S}_{n+1}(T)$; we call such a permutation in $\mathfrak{S}_{n+1}(T)$ a **son** of $\pi$.

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We say that the active sites of a permutation $\pi \in \mathfrak{S}_n(T)$ are **right justified** if the sites to the right of any active site are also active.
Theorem

Let $m \geq 3$ and $1 \leq p < m$. 
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1. $1 \in \mathfrak{S}_1(\Sigma^p_m)$ has two active sites and any $\pi \in \mathfrak{S}_n(\Sigma^p_m)$ has its active sites right justified.
Theorem

Let $m \geq 3$ and $1 \leq p < m$.

1. $1 \in \mathcal{S}_1(\Sigma_m^p)$ has two active sites and any $\pi \in \mathcal{S}_n(\Sigma_m^p)$ has its active sites right justified.

2. The number of active sites of the permutation $\sigma \in \mathcal{S}_{n+1}(\Sigma_m^p)$ obtained from $\pi \in \mathcal{S}_n(\Sigma_m^p)$ by inserting $n + 1$ into its $i$th active site does not depend on $\pi$ but only on $i$, and the number $k$ of active sites of $\pi$.

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Theorem

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   - \( i \), and
   - the number \( k \) of active sites of \( \pi \)

\[ \chi_{m,p}(k, i) \]  the number of active sites of \( \sigma \)
Theorem

\[ \chi_{m,p}(k, i) = \begin{cases} 
  k + 1 & \text{if } k < m - 1 \text{ or } k - m + p + 2 \leq i \leq k \\
  m - 1 & \text{if } k \geq m - 1 \text{ and } 1 \leq i \leq p \\
  m + i - p - 1 & \text{otherwise.} 
\end{cases} \]
Corollary

*The succession rule for the set of permutations $\mathcal{S}_n(\Sigma^p_m)$ is:*

root \quad (2)

rules \quad (k) \mapsto \\
\begin{cases} 
(k + 1)^k & \text{if } k < m - 1 \\
(m - 1)^p(m)(m + 1) \ldots (k)(k + 1)^{m-p-1} & \text{otherwise}
\end{cases}
Example

- \((m, p) = (3, 1)\) **Dyck rules**
  - root (2)
  - \((k) \mapsto (2)(3) \ldots (k)(k + 1)\)

- \((m, p) = (4, 1)\) **Schröder rules**
  - root (2)
  - rules (2) ↦ (3)(3)
  - \((k) \mapsto (3)(4) \ldots (k)(k + 1)(k + 1)\) if \(k \geq 3\)
Example

- \((m, p) = (4, 2)\) Grand Dyck rules
  - root \((2)\)
  - rules \((2) \Rightarrow (3)(3)\)
    \((k) \Rightarrow (3)(3)(4) \ldots (k)(k+1)\) if \(k \geq 3\)

- \((m, p) = (5, 2)\):
  - root \((2)\)
  - rules \((2) \Rightarrow (3)(3)\)
    \((3) \Rightarrow (4)(4)(4)\)
    \((k) \Rightarrow (4)(4)(5) \ldots (k)(k+1)(k+1)\) if \(k \geq 4\).
Any succession rule can be expressed as:

\[ a_\ell \rightarrow (\ell_1 \rightarrow v(u, 1) \cdot \ell_2 \rightarrow v(u, 2) \cdot \ell_3 \rightarrow v(u, 3) \cdot \ldots) \]

where \{\ell_1, \ell_2, \ldots\} is the set of admissible productions.

The ultimately zero integer sequence \{v(u, k)\}_{k \geq 0} gives the repetition order.

The matrix \( R = [u(i, j)]_{i, j \geq 1} \) is the production matrix of the succession rule.
Any succession rule can be expressed as:

- a root $\ell_1$
- and a set of productions

\[
\{(\ell_u) \leadsto (\ell_1)^v(u,1)(\ell_2)^v(u,2)(\ell_3)^v(u,3) \ldots \}u \geq 1
\]
Production matrices

- E. Deutsch, L. Ferrari, S. Rinaldi : 2005

Any succession rule can be expressed as:

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- and a set of productions

\[
\{(\ell_u) \sim (\ell_1)^{v(u,1)}(\ell_2)^{v(u,2)}(\ell_3)^{v(u,3)} \ldots \}_{u \geq 1}
\]

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\[
\{(\ell_u) \rightsquigarrow (\ell_1)^{\nu(u,1)}(\ell_2)^{\nu(u,2)}(\ell_3)^{\nu(u,3)} \ldots \}_{u \geq 1}
\]

where

- \( \{\ell_1, \ell_2, \ldots \} \) is the set of admissible
- the ultimately zero integer sequence \( \{\nu(u, k)\}_{k \geq 0} \) gives the repetition order

The matrix

\[
R = [u(i, j)]_{i, j \geq 1}
\]

is the \textit{production matrix} of the succession rule
**Example**

Dyck rule:

- root \((2)\)
- \((k) \rightsquigarrow (2)(3) \ldots (k)(k + 1)\)

Production matrix:

\[
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & \ldots \\
1 & 1 & 1 & 0 & 0 & 0 & \ldots \\
1 & 1 & 1 & 1 & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]
Example

Grand Dyck rule

- root (2)
- rules (2) $\rightsquigarrow$ (3)(3)
- $(k) \rightsquigarrow (3)(3)(4) \ldots (k)(k + 1)$ if $k \geq 3$

Production matrix:

$$
\begin{bmatrix}
0 & 2 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 2 & 1 & 0 & 0 & 0 & \ldots \\
0 & 2 & 1 & 1 & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix}
$$
The integer sequence corresponding to a succession rule (or equivalently, to a production matrix) is the sequence giving, for each $n$, the number of objects of size $n$ produced by the succession rule.
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$$\mathcal{G}_n(\Sigma^p_m)$$
root \quad (2) \\
rules \quad (k) \rightsquigarrow \\
\begin{cases} 
(k + 1)^k \text{ if } k < m - 1 \\
(m - 1)^p(m)(m + 1) \ldots (k)(k + 1)^{m-p-1} \text{ otherwise}
\end{cases}
root (2) rules (k) \rightsquigarrow
\begin{cases} 
(k + 1)^k & \text{if } k < m - 1 \\
(m - 1)^p(m)(m + 1)\ldots(k)(k + 1)^{m-p-1} & \text{otherwise}
\end{cases}

A_{m,p} =
\begin{bmatrix}
0 & 2 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 3 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 4 & \ldots & 0 & 0 & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\
0 & 0 & 0 & 0 & \ldots & m - 2 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & \ldots & p & m - p - 1 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & \ldots & p & 1 & m - p - 1 & 0 & \ldots \\
0 & 0 & 0 & 0 & \ldots & p & 1 & 1 & m - p - 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots 
\end{bmatrix}
For a production matrix $P$, $f_P$ is generating function of the integer sequence associated with $P$.

$u^\top$ is the row vector $(1 \ 0 \ 0 \ldots 0)$

$e$ is the vector $(1 \ 1 \ 1 \ldots 1)^\top$
Theorem (E. Deutsch, L. Ferrari, S. Rinaldi)

Let \( a, b, c \) be three nonnegative integers, \( P \) and \( Q \) two production matrices related by

\[
P = \begin{bmatrix} b & a \cdot u^\top \\ c \cdot e & Q \end{bmatrix}.
\]

Then

\[
f_P(x) = \frac{1 + axf_Q(x)}{1 - bx - acx^2f_Q(x)}.
\]
Corollary

Let $a$, $b$, $c$ be three positive integers and $P$ be an infinite production matrix of the form

$$P = \begin{bmatrix} b & a \cdot u^\top \\ c \cdot e & P \end{bmatrix}.$$ 

Then $f_P(x)$ satisfies the quadratic equation

$$acx^2 f_P(x)^2 - (1 - bx - ax)f_P(x) + 1 = 0.$$
Corollary

Let $a$ be an integer and $R$ a production matrix of the form

$$R = \begin{bmatrix}
1 & a & 0 & 0 & 0 & \ldots \\
1 & 1 & a & 0 & 0 & \ldots \\
1 & 1 & 1 & a & 0 & \ldots \\
1 & 1 & 1 & 1 & a & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots 
\end{bmatrix}.$$ 

Then

$$f_R(x) = \frac{N_a(x)}{2ax^2}$$

where

$$N_a(x) = 1 - (a + 1)x - \sqrt{1 + (a - 1)^2x^2 - 2(a + 1)x}.$$
Lemma

Let $P$ be a production matrix of the form

$$P = \begin{bmatrix}
    b & a & 0 & 0 & 0 & \ldots \\
    b & 1 & a & 0 & 0 & \ldots \\
    b & 1 & 1 & a & 0 & \ldots \\
    b & 1 & 1 & 1 & a & \ldots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.$$

Then

$$f_P(x) = \frac{2x + N_a(x)}{x(2 - 2bx - bN_a(x))}.$$
Main results

Lemma

\[ P = \begin{bmatrix}
0 & 2 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 3 & \ldots & 0 & 0 & 0 & 0 & \ldots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ldots & m-4 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & \ldots & 0 & m-3 & 0 & 0 & \ldots \\
0 & 0 & 0 & \ldots & 0 & 0 & m-2 & \ldots & \ddots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix} = Q \]

Then

\[ f_P(x) = \sum_{i=0}^{m-4} (i+1)! \cdot x^i + (m-2)! \cdot x^{m-3} \cdot f_Q(x). \]
Theorem

The generating function for the succession rule

root \ (2)\ 

rules \ (k) \ \rightsquigarrow \ \begin{cases} \quad (k + 1)^k \text{ if } k < m - 1 \\ \quad (m - 1)^p(m)(m + 1) \ldots (k)(k + 1)^{m-p-1} \end{cases}

is given by

\[ \Psi(x) = \sum_{i=0}^{m-4} \frac{(i+1)! \cdot x^i + (m-2)! \cdot x^{m-3} \cdot F(x)}{i!} , \]

where

\[ F(x) = \frac{2x + N_{m-p-1}(x)}{x(2 - 2px - pN_{m-p-1}(x))} . \]

and

\[ N_a(x) = 1 - (a + 1)x - \sqrt{1 + (a - 1)^2x^2 - 2(a + 1)x} . \]
Corollary

The generating function of the sequence \( \{ \text{card}(\mathfrak{S}_n(\Sigma^p_m)) \}_{n \geq 0} \) is \( x \cdot \Psi(x) \).
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Corollary

\[
\text{card}(\mathcal{G}_n(\Sigma^p_{p+1})) = \begin{cases} 
n! & \text{if } n < p - 1 \\ (p - 1)! \cdot p^{n-p+1} & \text{otherwise} \end{cases}
\]
Particular instances

- \( \text{card}(\mathcal{G}_n(\Sigma^1_5)) \)
  - first values: 0, 1, 2, 6, 24, 114, 600, 3372, 19824, \ldots
  - Sloane: A054872
  - generating function: \( x \cdot \left( 2 - 2x - \sqrt{1 - 8x + 4x^2} \right) \)

- \( \text{card}(\mathcal{G}_n(\Sigma^2_5)) \)
  - first values: 0, 1, 2, 6, 24, 108, 516, 2556, 12972 \ldots
  - generating function: \( x \cdot \left( 1 + 2x + 3x \cdot \frac{1-x-x\sqrt{1-6x+x^2}}{x+\sqrt{1-6x+x^2}} \right) \)

- \( \text{card}(\mathcal{G}_n(\Sigma^3_5)) \)
  - first values: 0, 1, 2, 6, 24, 102, 444, 1956, \ldots
  - generating function: \( x \cdot \left( 1 + 2x + 6x \cdot \frac{1-\sqrt{1-4x}}{-1+3\sqrt{1-4x}} \right) \)

- \( \text{card}(\mathcal{G}_n(\Sigma^4_5)) \)
  - first values: 0, 1, 2, 6, 24, 96, 384, 1536, 6144, \ldots
  - Sloane: A084509
  - generating function: \( x \cdot \frac{1-2x-2x^2}{1-4x} \)