Generalized Schröder permutations

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Notations

- $\mathfrak{S}_n$ is the set of length n permutations
- $\mathfrak{S}_n(A)$ is the set of permutations in $\mathfrak{S}_n$ avoiding each permutation in A
- $\mathfrak{S}(A) = \cup_{n=0}^{\infty} \mathfrak{S}_n(A)$

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$$\Gamma_m = \{\sigma \in \mathfrak{S}_m \mid \sigma(m-1) = m-1 \text{ and } \sigma(m) = m\}$$

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Theorem (Barcucci et al.)

The generating function of the sequence $(\mathrm{card}(\mathfrak{S}_n(\Gamma_m)))_{n\geq 0}$ *is*

$$\sum_{i=1}^{m-3} i! \cdot x^i + x^{m-4} \cdot (m-3)! \cdot \frac{1-(m-1)x-\sqrt{1-2(m-1)x+(m-3)^2x^2}}{2}$$

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For $1 \leq s, t \leq m$, $s \neq t$, define $\Gamma_{m;s,t} \subset \mathfrak{S}_m$ by

$$\Gamma_{m;s,t} = \{\sigma \in \mathfrak{S}_m \mid \sigma(s) = m - 1 \text{ and } \sigma(t) = m\}$$

In particular, $\Gamma_{m;m-1,m} = \Gamma_m$

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Theorem (Kremer 2000, 2003)

For

- $|t - s| \leq 2$, *or*
- $t \in \{1, m\}$

the cardinality of $\mathfrak{S}_n(\Gamma_{m;s,t})$ does not depend on s and t

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For $1 \leq j < m$ let define $\Sigma_{m,j} \subset \mathfrak{S}_m$ by:

$\sigma \in \Sigma_{m,j}$ iff

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Example

- $\Sigma_{4,1} = \{\mathbf{4123}, \mathbf{4213}\}$
- $\Sigma_{4,2} = \{\mathbf{4132}, \mathbf{4231}\}$

For $1 \leq p < m$ define $\Sigma_m^p \subset \mathfrak{S}_m$ by

$$\Sigma_m^p = \bigcup_{j=1}^{p} \Sigma_{m,j}$$

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  $\Sigma_2^1 = \{\mathbf{21}\}$ and $\mathrm{card}(\mathfrak{S}_n(\Sigma_2^1)) = 1$

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- $(m, p) = (4, 1)$
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- $(m, p) = (4, 2)$
  $\Sigma_4^2 = \{\mathbf{4}12\mathbf{3}, \mathbf{4}21\mathbf{3}, \mathbf{4}1\mathbf{3}2, \mathbf{4}2\mathbf{3}1\}$ and $\mathrm{card}(\mathfrak{S}_n(\Sigma_4^2))$ is the $(n-1)$th central binomial coefficient $\binom{2n-2}{n-1}$

$$\mathrm{card}(\Sigma_m^p)$$

$m \backslash p$	1	2	3	4
2	1	–	–	–
3	Catalan	2^{n-1}	–	–
4	Schröder	$\binom{2n-2}{n-1}$	$2 \cdot 3^{n-2}$ A025192	–
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We give a generating function for the set

$$\mathfrak{S}(\Sigma_m^p)$$

Tools

- Generating trees
- Production matrices

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- Generating trees: how $\mathfrak{S}(\Sigma_m^p)$ can be recursively defined?
- Production matrices: how this definition can be turned into a generating function?

Generating trees

A *succession* (or *ECO*) *rule* is a formal system consisting of a root e_0 (or axiom) and a set of *productions* of the form

$$(k) \leadsto (e_1(k))(e_2(k)) \ldots (e_k(k))$$

Succession rule explains how an object of size n can be uniquely expanded into several objects of size $n+1$.

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- For a permutation $\pi \in \mathfrak{S}_n(T)$, i is an *active site* if the permutation obtained from π by inserting $n+1$ into its ith site is a permutation in $\mathfrak{S}_{n+1}(T)$; we call such a permutation in $\mathfrak{S}_{n+1}(T)$ a *son* of π

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- For any $\pi \in \mathfrak{S}_n(T)$, by erasing n in π one obtains a permutation in $\mathfrak{S}_{n-1}(T)$; or equivalently, any permutation in $\mathfrak{S}_n(T)$ is obtained from a permutation in $\mathfrak{S}_{n-1}(T)$ by inserting n in one of its active sites

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- We say that the active sites of a permutation $\pi \in \mathfrak{S}_n(T)$ are *right justified* if the sites to the right of any active site are also active.

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2. *The number of active sites of the permutation* $\sigma \in \mathfrak{S}_{n+1}(\Sigma_m^p)$ *obtained from* $\pi \in \mathfrak{S}_n(\Sigma_m^p)$ *by inserting* $n+1$ *into its* ith *active site does not depend on* π *but only on*
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$\chi_{m,p}(k,i)$ the number of active sites of σ

Theorem

$$\chi_{m,p}(k,i) = \begin{cases} k+1 & \text{if } k < m-1 \text{ or} \\ & k-m+p+2 \le i \le k \\ m-1 & \text{if } k \ge m-1 \text{ and } 1 \le i \le p \\ m+i-p-1 & \text{otherwise.} \end{cases}$$

Corollary

The succession rule for the set of permutations $\mathfrak{S}_n(\Sigma_m^p)$ *is:*

root (2)

rules $(k) \rightsquigarrow$

$$\begin{cases} (k+1)^k \text{ if } k < m-1 \\ (m-1)^p(m)(m+1)\dots(k)(k+1)^{m-p-1} \text{ otherwise} \end{cases}$$

Example

- $(m,p) = (3,1)$ Dyck rules

$$\begin{array}{lll} \text{root} & (2) & \\ & (k) & \rightsquigarrow \quad (2)(3)\dots(k)(k+1) \end{array}$$

- $(m,p) = (4,1)$ Schröder rules

$$\begin{array}{llll} \text{root} & (2) & & \\ \text{rules} & (2) & \rightsquigarrow & (3)(3) \\ & (k) & \rightsquigarrow & (3)(4)\dots(k)(k+1)(k+1) \quad \text{if } k \geq 3 \end{array}$$

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- $(m, p) = (4, 2)$ Grand Dyck rules

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- $(m, p) = (5, 2)$:

$$\begin{array}{llll} \text{root} & (2) & & \\ \text{rules} & (2) & \rightsquigarrow & (3)(3) \\ & (3) & \rightsquigarrow & (4)(4)(4) \\ & (k) & \rightsquigarrow & (4)(4)(5)\dots(k)(k+1)(k+1) \quad \text{if } k \geq 4. \end{array}$$

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- E. Deutsch, L. Ferrari, S. Rinaldi : 2005

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Any succession rule can be expressed as:

- a root ℓ_1
- and a set of productions

$$\{(\ell_u) \rightsquigarrow (\ell_1)^{v(u,1)}(\ell_2)^{v(u,2)}(\ell_3)^{v(u,3)} \dots\}_{u \geq 1}$$

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where

- $\{\ell_1, \ell_2, \ldots\}$ is the set of admissible
- the ultimately zero integer sequence $\{v(u,k)\}_{k \geq 0}$ gives the repetition order

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The matrix

$$R = [u(i,j)]_{i,j \geq 1}$$

is the *production matrix* of the succession rule

Example

Dyck rule:

$$\begin{array}{lll} \text{root} & (2) & \\ & (k) & \rightsquigarrow (2)(3)\dots(k)(k+1) \end{array}$$

production matrix:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 1 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Example

Grand Dyck rule

$$\begin{array}{llll} \text{root} & (2) & & \\ \text{rules} & (2) & \rightsquigarrow & (3)(3) \\ & (k) & \rightsquigarrow & (3)(3)(4)\dots(k)(k+1) \quad \text{if } k \geq 3 \end{array}$$

production matrix:

$$\begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 & \dots \\ 0 & 2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 1 & 1 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

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$$\mathfrak{S}_n(\Sigma_m^p)$$

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$$\begin{cases} (k+1)^k \text{ if } k < m-1 \\ (m-1)^p(m)(m+1)\ldots(k)(k+1)^{m-p-1} \text{ otherwise} \end{cases}$$

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$$\begin{cases} (k+1)^k \text{ if } k < m-1 \\ (m-1)^p(m)(m+1)\dots(k)(k+1)^{m-p-1} \text{ otherwise} \end{cases}$$

$$A_{m,p} = \begin{bmatrix} 0 & 2 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 3 & 0 & \dots & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 4 & \dots & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & 0 & \dots & m-2 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots & p & m-p-1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots & p & 1 & m-p-1 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots & p & 1 & 1 & m-p-1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

For a production matrix P, f_P is generating function of the integer sequence associated with P

$u^\top$ is the row vector $(1\,0\,0\,\ldots 0)$

e is the vector $(1\,1\,1\,\ldots 1)^\top$

Theorem (E. Deutsch, L. Ferrari, S. Rinaldi)

Let a, b, c be three nonnegative integers, P and Q two production matrices related by

$$P = \begin{bmatrix} b & a \cdot u^\top \\ c \cdot e & Q \end{bmatrix}.$$

Then

$$f_P(x) = \frac{1 + axf_Q(x)}{1 - bx - acx^2 f_Q(x)}.$$

Corollary

Let a, b, c be three positive integers and P be an infinite production matrix of the form

$$P = \left[\begin{array}{cc} b & a \cdot u^{\top} \\ c \cdot e & P \end{array} \right].$$

Then $f_P(x)$ satisfies the quadratic equation

$$acx^2 f_P(x)^2 - (1 - bx - ax) f_P(x) + 1 = 0.$$

Corollary

Let a be an integer and R a production matrix of the form

$$R = \begin{bmatrix} 1 & a & 0 & 0 & 0 & \dots \\ 1 & 1 & a & 0 & 0 & \dots \\ 1 & 1 & 1 & a & 0 & \dots \\ 1 & 1 & 1 & 1 & a & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then

$$f_R(x) = \frac{N_a(x)}{2ax^2}$$

where

$$N_a(x) = 1 - (a+1)x - \sqrt{1 + (a-1)^2x^2 - 2(a+1)x}.$$

Lemma

Let P be a production matrix of the form

$$P = \begin{bmatrix} b & a & 0 & 0 & 0 & \dots \\ b & 1 & a & 0 & 0 & \dots \\ b & 1 & 1 & a & 0 & \dots \\ b & 1 & 1 & 1 & a & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then

$$f_P(x) = \frac{2x + N_a(x)}{x(2 - 2bx - bN_a(x))}$$

Main results

Lemma

$$P = \begin{bmatrix} 0 & 2 & 0 & \ldots & 0 & 0 & 0 & \ldots \\ 0 & 0 & 3 & \ldots & 0 & 0 & 0 & \ldots \\ & & & \ddots & & & & \\ 0 & 0 & 0 & \ldots & m-4 & 0 & 0 & \ldots \\ 0 & 0 & 0 & \ldots & 0 & m-3 & 0 & \ldots \\ 0 & 0 & 0 & \ldots & 0 & 0 & m-2 & \ldots \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & & Q \end{bmatrix}$$

Then

$$f_P(x) = \sum_{i=0}^{m-4} (i+1)! \cdot x^i + (m-2)! \cdot x^{m-3} \cdot f_Q(x).$$

Theorem

The generating function for the succession rule

$$\begin{array}{lll} \text{root} & (2) & \\ \text{rules} & (k) & \rightsquigarrow \begin{cases} (k+1)^k \text{ if } k < m-1 \\ (m-1)^p (m)(m+1)\dots(k)(k+1)^{m-p-1} \end{cases} \end{array}$$

is given by

$$\Psi(x) = \sum_{i=0}^{m-4} (i+1)! \cdot x^i + (m-2)! \cdot x^{m-3} \cdot F(x),$$

where

$$F(x) = \frac{2x + N_{m-p-1}(x)}{x(2 - 2px - pN_{m-p-1}(x))}.$$

and

$$N_a(x) = 1 - (a+1)x - \sqrt{1 + (a-1)^2 x^2 - 2(a+1)x}.$$

Corollary

The generating function of the sequence $\{\mathrm{card}(\mathfrak{S}_n(\Sigma_m^p))\}_{n\geq 0}$ *is* $x \cdot \Psi(x)$.

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Corollary

$$\mathrm{card}(\mathfrak{S}_n(\Sigma_{p+1}^p)) = \begin{cases} n! & \text{if } n < p-1 \\ (p-1)! \cdot p^{n-p+1} & \text{otherwise} \end{cases}$$

Particular instances

- $\text{card}(\mathfrak{S}_n(\Sigma_5^1))$

  first values: 0, 1, 2, 6, 24, 114, 600, 3372, 19824, ...
  Sloane: `A054872`
  generating function: $x \cdot \left(2 - 2x - \sqrt{1 - 8x + 4x^2}\right)$

- $\text{card}(\mathfrak{S}_n(\Sigma_5^2))$

  first values: 0, 1, 2, 6, 24, 108, 516, 2556, 12972 ...
  generating function: $x \cdot \left(1 + 2x + 3x \cdot \frac{1-x-\sqrt{1-6x+x^2}}{x+\sqrt{1-6x+x^2}}\right)$

- $\text{card}(\mathfrak{S}_n(\Sigma_5^3))$

  first values: 0, 1, 2, 6, 24, 102, 444, 1956, ...
  generating function: $x \cdot \left(1 + 2x + 6x \cdot \frac{1-\sqrt{1-4x}}{-1+3\sqrt{1-4x}}\right)$

- $\text{card}(\mathfrak{S}_n(\Sigma_5^4))$

  first values: 0, 1, 2, 6, 24, 96, 384, 1536, 6144, ...
  Sloane: `A084509`
  generating function: $x \cdot \frac{1-2x-2x^2}{1-4x}$