More restricted growth functions: Gray codes and exhaustive generations

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Outline

- Some definitions:
  - set partitions, restricted (bounded) growth functions
  - Gray codes
  - generating algorithms
  - order relations
- Main results:
  - Gray codes
  - generating algorithms

  for bounded growth functions

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Set partitions

A *set partition* of $[n] = \{1, 2, \ldots, n\}$ is a collection

$$D_0, D_1, \ldots, D_{k-1}$$

of disjoint subsets (blocks) of $[n]$ whose union is $[n]$

A partition of $[n]$ is in *standard form* if

$$\min D_0 < \min D_1 < \cdots < \min D_{k-1}$$

and in this case the partition is denoted

$$D_0 / D_1 / \cdots / D_{k-1}$$

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The number of partitions of a set of cardinality n is B_n, the nth *Bell number*

The number of partitions of a set of cardinality n, into k nonempty subset is $S_{n,k}$, the *Stirling numbers of the second kind*:

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Restrictive growth functions

A *restricted growth function* of length n is an integer sequence $s = s_1 s_2 \cdots s_n$ such that

$s_1 = 0$, and

$0 \leq s_{i+1} \leq \max\{s_1, \ldots, s_i\} + 1$, for all $1 \leq i \leq n-1$

There is a bijection between the set of restricted growth functions of length n and the set of partitions of $[n]$, namely:

- $$s_1 s_2 \cdots s_n \mapsto D_0/D_1/\cdots/D_{k-1}$$
  if and only if $s_j = i$ implies $j \in D_i$; or, conversely,
- $$D_0/D_1/\cdots/D_{k-1} \mapsto s_1 s_2 \cdots s_n$$
  if and only if $j \in D_i$ implies $s_j = i$.

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Example

$\{\{2,5\},\{6\},\{4,7\},\{1,3,8\}\}$ is apartition of $\{1,2,\ldots,8\}$

$\{\{1,3,8\},\{2,5\},\{4,7\},\{6\}\}$ its standard form

$1,3,8/2,5/4,7/6$

$\underbrace{1,3,8}_{D_0}/\underbrace{2,5}_{D_1}/\underbrace{4,7}_{D_2}/\underbrace{6}_{D_3}$

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Bounded restricted growth functions

For an integer $b > 0$, $s = s_1 s_2 \dots s_n$ is a *b-bounded restricted growth function* if

$$s_i \leq b \text{ for all } 1 \leq i \leq n$$

$R_n(b)$ denotes the set of *b-bounded* sequences in R_n

$$R_n(b) = \{s_1 s_2 \dots s_n \in R_n \ : \ \max\{s_i\}_{i=1}^n \leq b\}.$$

$R_n(b)$ is in bijection with the partitions of the set $[n]$, into at most $b+1$ nonempty subset and

$$\text{card } R_n(b) = \sum_{k=1}^{b+1} S_{n,k}$$

$$P_n(b) = \{s_1 s_2 \dots s_n \in R_n \ : \ \max\{s_i\}_{i=1}^n = b\}.$$

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The set $R_5(2)$

0 0 0 0 0	0 1 0 0 0	0 1 1 1 2
0 0 0 0 1	0 1 0 0 1	0 1 1 2 2
0 0 0 1 0	0 1 0 0 2	0 1 1 2 1
0 0 0 1 1	0 1 0 1 0	0 1 1 2 0
0 0 0 1 2	0 1 0 1 1	0 1 2 2 0
0 0 1 0 0	0 1 0 1 2	0 1 2 2 1
0 0 1 0 1	0 1 0 2 2	0 1 2 2 2
0 0 1 0 2	0 1 0 2 1	0 1 2 1 2
0 0 1 1 0	0 1 0 2 0	0 1 2 1 1
0 0 1 1 1	0 1 1 0 0	0 1 2 1 0
0 0 1 1 2	0 1 1 0 1	0 1 2 0 2
0 0 1 2 2	0 1 1 0 2	0 1 2 0 1
0 0 1 2 1	0 1 1 1 0	0 1 2 0 0
0 0 1 2 0	0 1 1 1 1	

Gray codes

A *Gray code* for a combinatorial class is a listing of its objects in which only *“small change”* takes place between any two consecutive objects

A *d-Gray code* is a Gray code such that the Hamming distance between any two consecutive objects is at most d.

Known Gray codes for

- permutations: Steinhaus-Johnson-Trotter (1962-1964)
- involutions: Walsh (2001)
- derangements: Baril-Vajnovszki (2004)
- etc.

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Generating algorithms

We we present Gray codes and constant amortized time (CAT) algorithm for generating these Gray codes for

- $R_n(b)$
- $P_n(b)$, b odd

Some previous works for generating R_n in Gray code:

- Knuth (1975)
- Ruskey (improvement of Knuth's algorithm)
- Ruskey and Savage: a loop-free implementation (1984)

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Order relations

The *lexicographic order* on $\{0, 1, \ldots, m-1\}^n$ is defined as:

$$s_1 s_2 \ldots s_n < t_1 t_2 \ldots t_n,$$

if

$$s_k < t_k$$

where k is the leftmost position where s and t differ.

Definition

The *Reflected Gray Code order* on $\{0, 1, \ldots, m-1\}^n$ is defined as:

$$s_1 s_2 \ldots s_n \prec t_1 t_2 \ldots t_n,$$

if either

- $\sum_{i=1}^{k-1} s_i$ is even and $s_k < t_k$, or
- $\sum_{i=1}^{k-1} s_i$ is odd and $s_k > t_k$,

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Definition

The *co-Reflected Gray Code order* on $\{0, 1, \ldots, m-1\}^n$ is defined as:

$$s_1 s_2 \ldots s_n \lhd t_1 t_2 \ldots t_n,$$

if either

- u_k is even and $s_k < t_k$, or
- u_k is odd and $s_k > t_k$,

where k is the leftmost position where s and t differ, and

$$u_k = \sum_{i=1}^{k-1} [s_i \neq 0 \text{ and } s_i \text{ is even}],$$

and $[\cdot]$ is the Iverson bracket.

$u_k =$ the number of non-zero even symbols in $s_1 s_2 \ldots s_{k-1}$

Definition

The *co-Reflected Gray Code order* on $\{0, 1, \ldots, m-1\}^n$ is defined as:

$$s_1 s_2 \ldots s_n \triangleleft t_1 t_2 \ldots t_n,$$

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Example: The set $\{0, 1, 2\}^3$ listed in $\triangleleft$ order

0 0 0	1 0 0	2 2 0
0 0 1	1 0 1	2 2 1
0 0 2	1 0 2	2 2 2
0 1 0	1 1 0	2 1 2
0 1 1	1 1 1	2 1 1
0 1 2	1 1 2	2 1 0
0 2 2	1 2 2	2 0 2
0 2 1	1 2 1	2 0 1
0 2 0	1 2 0	2 0 0

The main results

Theorem

For any $n, b \geq 1$ and b odd, $R_n(b)$ listed in $\prec$ order is a 3-Gray code.

Proposition

Let

- $b \geq 2$ *and odd*
- $a = a_1 a_2 \ldots a_k$, $k < n$

If s is the $\prec$*-last (resp. the* $\prec$*-first) sequence in* $R_n(b)$ *with the prefix a, then s has one of these forms:*

- $s = aM0\ldots0$ *if* $\sum_{i=1}^{k-1} s_i$ *is even (resp. odd) and M is odd;*
- $s = aM(M+1)0\ldots0$ *if* $\sum_{i=1}^{k-1} s_i$ *is even (resp. odd) and M is even;*
- $s = a0\ldots0$ *if* $\sum_{i=1}^{k-1} s_i$ *is odd (resp. even),*

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For any $n \geq 1$, $b \geq 2$ and even, $R_n(b)$ listed in $\lhd$ order is a 3-Gray code.

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Corollary

For any $n \geq 1$, R_n listed in both $\prec$ and $\triangleleft$ order are 3-Gray codes.

Theorem

For any $n, b \geq 1$, b odd, $P_n(b)$ listed in $\prec$ order is a 5-Gray code.

procedure GEN1(k, dir, M: integer)
global s, n, $bound$: integer;
local i, u: integer;
if $M = bound$ **then** $M := bound - 1$;
if $k = n + 1$ **then** TYPE();
else if dir is even **then**
  for $i := 0$ **to** $M + 1$
    $s_k := i$;
    if $M < s_k$ **then** $u := s_k$; **else** $u := M$;
    GEN1($k + 1, i, u$);
 else for $i := M + 1$ **downto** 0
    $s_k := i$;
    if $M < s_k$ **then** $u := s_k$; **else** $u := M$;
    GEN1($k + 1, i + 1, u$);

Generating algorithm for $R_n(b)$, $b \geq 1$ and odd, in RGC order

```
procedure GEN2(k, dir, M: integer)
global s, n, bound: integer;
local i, u: integer;
if M + 1 > bound then M := bound − 1;
if k = n + 1 then TYPE();
else if dir is even then
        for i := 0 to M + 1
            s_k := i;
            if M < s_k then u := s_k; else u := M;
            if s_k = 0 then GEN2(k + 1, 0, u);
            else GEN2(k + 1, i + 1, u);
      else for i := M + 1 downto 0
            b_k := i;
            if M < s_k then u := s_k; else u := M;
            if  s_k = 0 then GEN2(k + 1, 1, u);
            else GEN2(k + 1, i, u);
```

Generating algorithm for $R_n(b)$, $b \geq 2$ and even, in co-RGC

```
procedure GEN3(k, dir, M, flag: integer)
if k = n + 1 then TYPE();
else if bound − M = n − k + 1 and flag = 0 then
    Assign unique values for s_k ... s_n;
    TYPE();
else if M = bound then M := M − 1;flag := 1;
    if dir is even then
      for i := 0 to M + 1
          s_k := i;
          if M < s_k then u := s_k; else u := M;
          GEN3(k + 1, i, u, flag);
    else for i = M + 1 downto 0
          s_k := i;
          if M < s_k then u := s_k; else u := M;
          GEN3(k + 1, i + 1, u, flag);
```

Generating algorithm for $P_n(b)$, $b \geq 1$ and odd, with respect to RGC order

Example

The set $R_5(2)$ isted in $\lhd$ is

0 0 0 0 0	0 1 0 0 0	0 1 1 1 2
0 0 0 0 1	0 1 0 0 1	0 1 1 2 2
0 0 0 1 0	0 1 0 0 2	0 1 1 2 1
0 0 0 1 1	0 1 0 1 0	0 1 1 2 0
0 0 0 1 2	0 1 0 1 1	0 1 2 2 0
0 0 1 0 0	0 1 0 1 2	0 1 2 2 1
0 0 1 0 1	0 1 0 2 2	0 1 2 2 2
0 0 1 0 2	0 1 0 2 1	0 1 2 1 2
0 0 1 1 0	0 1 0 2 0	0 1 2 1 1
0 0 1 1 1	0 1 1 0 0	0 1 2 1 0
0 0 1 1 2	0 1 1 0 1	0 1 2 0 2
0 0 1 2 2	0 1 1 0 2	0 1 2 0 1
0 0 1 2 1	0 1 1 1 0	0 1 2 0 0
0 0 1 2 0	0 1 1 1 1	

Thank you !

ありがとう