Generalized Schröder permutations

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Abstract

We give the generating function for the integer sequence enumerating a class of pattern avoiding permutations depending on two parameters: m and p. The avoided patterns are the permutations of length m with the largest element in the first position and the second largest in one of the last p positions. For particular instances of m and p we obtain pattern avoiding classes enumerated by Schröder, Catalan and central binomial coefficient numbers, and thus, the obtained two-parameter generating function gathers known generating functions under one roof and expresses new ones. This work generalizes some earlier results of Barcucci *et al.* (2000) and Kremer (2000, 2003).

1 Introduction

Pattern avoiding permutations have become a very active research area mainly since the first systematic study published by Simion and Schmidt in 1985 [1]. This is in part due to many restricted classes of permutations being in bijection with well-known combinatorial structures, and so, their study allows to re-express known results in terms of pattern avoidance and state new ones. In this paper we present a two-parameter generating function for *generalized Schröder permutations*, which simultaneously generalize permutation classes counted by Schröder, Catalan and central binomial coefficient numbers. A few particular instances of this generating function correspond to known integer sequences, some of them not previously related to permutation classes, see Table 1.

Let $\mathfrak{S}_n$ be the set of length n permutations. For two permutations $\sigma \in \mathfrak{S}_k$ and $\pi \in \mathfrak{S}_n$ we say that π *avoids* σ if there is no sequence $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $\pi_{i_1}\pi_{i_2}\ldots\pi_{i_k}$ is order-isomorphic to σ. In this context σ is called *pattern* and for a set of patterns A, $\mathfrak{S}_n(A)$ denotes the set of permutations in $\mathfrak{S}_n$ avoiding each pattern in A, and $\mathfrak{S}(A) = \cup_{n=0}^{\infty}\mathfrak{S}_n(A)$.

For an integer $m \geq 2$, define $\Gamma_m \subset \mathfrak{S}_m$ by

$$\Gamma_m = \{\sigma \in \mathfrak{S}_m \mid \sigma(m-1) = m-1 \text{ and } \sigma(m) = m\}.$$

In other words, Γ_m is the set of length m permutations with fixed points in the last and last but one position.

Example 1.

- $\Gamma_3 = \{1\mathbf{23}\}$ and so $\mathrm{card}(\mathfrak{S}_n(\Gamma_3)) = c_n$, the nth Catalan number,
- $\Gamma_4 = \{12\mathbf{34}, 21\mathbf{34}\}$ and so $\mathrm{card}(\mathfrak{S}_n(\Gamma_4)) = r_n$, the nth Schröder number.

In [2] Barcucci *et al.* gave a multivariate generating function for the set of permutations in $\mathfrak{S}(\Gamma_m)$ with the parameters: length, left minima and non-inversions. In particular, the generating function of the sequence $\{\mathrm{card}(\mathfrak{S}_n(\Gamma_m))\}_{n\geq 0}$ is

$$\sum_{i=1}^{m-3} i!x^i + x^{m-4}(m-3)!\frac{1-(m-1)x-\sqrt{1-2(m-1)x+(m-3)^2x^2}}{2}. \tag{1}$$

For three integers $1 \leq s, t \leq m$, $s \neq t$, define $\Gamma_{m;s,t} \subset \mathfrak{S}_m$ by

$$\Gamma_{m;s,t} = \{\sigma \in \mathfrak{S}_m \mid \sigma(s) = m-1 \text{ and } \sigma(t) = m\}$$

and, in particular, $\Gamma_{m;m-1,m} = \Gamma_m$. In [5, 6] Kremer gave the following result.

Theorem 1 ([5, 6])**.** *With the notation above, for $|s-t| \leq 2$, or $t \in \{1, m\}$ the cardinality of $\mathfrak{S}_n(\Gamma_{m;s,t})$ does not depend on s and t.*

This theorem implies that, under the above conditions on s and t, the generating function of the sequence $\{\mathrm{card}(\mathfrak{S}_n(\Gamma_{m;s,t}))\}_{n\geq 0}$ is given in (1).

In this paper we generalize these results by imposing that the second largest element of the length m forbidden patterns occurs in *one of the last p positions*. Formally, let m and j be two integers, $1 \leq j < m$, and define $\Sigma_{m,j} \subset \mathfrak{S}_m$ by

$$\Sigma_{m,j} = \{\sigma \in \mathfrak{S}_m \mid \sigma(1) = m \text{ and } \sigma(m+1-j) = m-1\}.$$

For example, $\Sigma_{4,1} = \{\mathbf{4}12\mathbf{3}, \mathbf{4}21\mathbf{3}\}$ and $\Sigma_{4,2} = \{\mathbf{4}1\mathbf{3}2, \mathbf{4}2\mathbf{3}1\}$.

Now, for $1 \leq p < m$ define $\Sigma_m^p \subset \mathfrak{S}_m$ by

$$\Sigma_m^p = \bigcup_{j=1}^{p} \Sigma_{m,j},$$

and, for instance, $\Sigma_4^2 = \Sigma_{4,1} \cup \Sigma_{4,2} = \{\mathbf{4}12\mathbf{3}, \mathbf{4}21\mathbf{3}, \mathbf{4}1\mathbf{3}2, \mathbf{4}2\mathbf{3}1\}$.

Example 2.

- $\Sigma_2^1 = \{\mathbf{21}\}$ and so $\mathrm{card}(\mathfrak{S}_n(\Sigma_2^1)) = 1$.
- $\Sigma_3^1 = \{\mathbf{3}1\mathbf{2}\}$ and so $\mathfrak{S}_n(\Sigma_3^1)$ is counted with the Catalan number.
- $\Sigma_3^2 = \{\mathbf{3}1\mathbf{2}, \mathbf{32}1\}$ and $\mathrm{card}(\mathfrak{S}_n(\Sigma_3^2)) = 2^{n-1}$, see [1].
- $\Sigma_4^1 = \{\mathbf{4}12\mathbf{3}, \mathbf{4}21\mathbf{3}\}$ and $\mathfrak{S}_n(\Sigma_4^1)$ is counted by the Schröder number, see for instance [4, 7]. The sets $\mathfrak{S}_n(\Sigma_4^1)$ for $n = 1, \ldots, 4$ are given in Figure 1.
- $\Sigma_4^2 = \{\mathbf{4}12\mathbf{3}, \mathbf{4}21\mathbf{3}, \mathbf{4}1\mathbf{3}2, \mathbf{4}2\mathbf{3}1\}$ and $\mathfrak{S}_n(\Sigma_4^2)$ is counted by the $(n-1)$th central binomial coefficient $\binom{2n-2}{n-1}$, see [4].

2 Generating trees

A *succession* (or *ECO*) *rule* is a formal system consisting of a root e_0 (or axiom) and a set of *productions* of the form

$$(k) \rightsquigarrow (e_1(k))(e_2(k))\dots(e_k(k)) \tag{2}$$

where e_0 and each $e_i(k)$, $1 \le i \le k$, are integers. The right side of these productions are sequences of parenthesed integers. A succession rule explains how an object of size n can be uniquely expanded into several objects of size $n+1$. Note that in productions above the size of objects does not occur explicitly.

Now we explain the succession rule techniques in the context of pattern avoidance. The *sites* of $\pi \in \mathfrak{S}_n$ are the positions between two consecutive entries, before the first and after the last entry; and they are numbered, from right to left, from 1 to $n+1$. For a permutation $\pi \in \mathfrak{S}_n(T)$, with T a set of forbidden patterns, i is an *active site* if the permutation obtained from π by inserting $n+1$ into its ith site is a permutation in $\mathfrak{S}_{n+1}(T)$; we call such a permutation in $\mathfrak{S}_{n+1}(T)$ a *son* of π. For any $n > 1$ and $\pi \in \mathfrak{S}_n(T)$, by erasing n in π one obtains a permutation in $\mathfrak{S}_{n-1}(T)$; or equivalently, any permutation in $\mathfrak{S}_n(T)$ is obtained from a permutation in $\mathfrak{S}_{n-1}(T)$ by inserting n in one of its active sites. We say that the active sites of a permutation $\pi \in \mathfrak{S}_n(T)$ are *right justified* if the sites to the right of any active site are also active. See Figure 1 for an example.

Define Θ_m^p to be the set of permutations which are length $(m-1)$ suffixes of permutations in Σ_m^p. In other words, Θ_m^p is the set of permutations θ in $\mathfrak{S}_{m-1}$ with $m-1 \in \{\theta(m-p), \theta(m-p+1), \dots, \theta(m-1)\}$. Permutations in Θ_m^p are critical in our construction of a generating tree for $\mathfrak{S}_n(\Sigma_m^p)$ since they are 'precursors' of patterns in Σ_m^p. Indeed, the insertion of $n+1$ into a site of $\pi \in \mathfrak{S}_n(\Sigma_m^p)$ produces an occurrence of a pattern in Σ_m^p if and only if a pattern belonging to Θ_m^p occurs in π on the right of this site. For short, in a permutation π, an occurrence of a pattern in Θ_m^p will be called a Θ-*pattern*.

Lemma 1. *Let $m \ge 3$ and $1 \le p < m$. The length one permutation $1 \in \mathfrak{S}_1(\Sigma_m^p)$ has two active sites and any $\pi \in \mathfrak{S}_n(\Sigma_m^p)$ has its active sites right justified.*

Proof. For $m \ge 3$, it is clear that both permutations 12 and 21 belong to $\mathfrak{S}_2(\Sigma_m^p)$ and so $1 \in \mathfrak{S}_1(\Sigma_m^p)$ has two active sites.

Now let us suppose that $\pi \in \mathfrak{S}_n(\Sigma_m^p)$ has at least a non-active site, and let i be the rightmost of them. That is, the site between the entries π_{n-i+1} and π_{n-i+2} is the rightmost non-active site of π. It follows that the suffix $\pi_{n-i+2}\pi_{n-i+3}\dots\pi_n$ contains a Θ-pattern, and so does any longer suffix. Thus all the sites to the left of i are non-active. $\square$

Lemma 2. *If π is a permutation in $\mathfrak{S}_n(\Sigma_m^p)$ with k active sites and $k < m-1$, then each permutation obtained from π by inserting $n+1$ in any active site of π yields a permutation (in $\mathfrak{S}_{n+1}(\Sigma_m^p)$) with $k+1$ active sites.*

Proof. First remark that, in a permutation $\pi \in \mathfrak{S}_n(\Sigma_m^p)$ with $n \ge m-2$, the rightmost $m-1$ sites are active. Indeed the insertion of $(n+1)$ into the $(m-1)$th (right to left) active site can not produces a pattern in Σ_m^p. It results that if $\pi \in \mathfrak{S}_n(\Sigma_m^p)$ has k active sites, with $k < m-1$, then $n = k-1 < m-2$. Thus the insertion of $n+1$ in any active site of π produces a permutation with $k+1 = n+2$ active sites. $\square$

Theorem 2. *The succession rule for the set of permutations $\mathfrak{S}_n(\Sigma_m^p)$ is:*

$$\begin{array}{lll} \text{root} & (2) & \\ \text{rules} & (k) & \rightsquigarrow \left\{ \begin{array}{ll} (k+1)^k & \text{if } k < m-1, \\ (m-1)^p(m)(m+1)\dots(k)(k+1)^{m-p-1} & \text{if } k \geq m-1. \end{array} \right. \end{array}$$

Proof. When $k < m-1$ the derivation $(k) \rightsquigarrow (k+1)^k$ follows from Lemma 2.
Let $\pi \in \mathfrak{S}_n(\Sigma_m^p)$ be a permutation with $k \geq m-1$ active sites. Clearly, the length of π is at least $k-1$, and the length $k-1$ suffix of π does not contain any Θ-pattern. For an active site $i \in \{1, 2, \dots, k\}$ let σ be the permutation obtained from π by inserting $n+1$ in the ith (from right to left) active site of π. We will distinguish three cases.

- If $i \in \{1, 2, \dots, p\}$, then the length $m-1$ suffix of σ is a Θ-pattern and σ has $m-1$ active sites.
- If $k > m-1$, then the set $\{p+1, p+2, \dots, p+k-m+1\}$ is not empty, and let i belong to it. The length $(m-p+i-1)$ suffix of σ contains a Θ-pattern and no shorter suffix of σ contains such a pattern. In this case σ has $(m-p+i-1)$ active sites. In particular, for $i = p+1$, σ has m active sites; for $i = p+2$, σ has $m+1$; $\dots$; for $i = p+k-m+1$, σ has k active sites.
- If $i \in \{p+k-m+2, p+k-m+3, \dots, k\}$, then the length k suffix of σ does not contain any Θ-pattern and all the rightmost $k+1$ sites of σ are active.

Combining these three cases we obtain the derivation $(k) \rightsquigarrow (m-1)^p(m)(m+1)\dots(k)(k+1)^{m-p-1}$ when $k \geq m-1$.

□

Remark 1. *Let $m \geq 3$ and p, $1 \leq p < m$. The number of active sites of the permutation $\sigma \in \mathfrak{S}_{n+1}(\Sigma_m^p)$ obtained from $\pi \in \mathfrak{S}_n(\Sigma_m^p)$ by inserting $n+1$ into its* ith *active site does not depend on π but only on i and on the number k of active sites of π.*

Example 3. The succession rules in Theorem 2 becomes:

- Dyck rules for $(m,p) = (3,1)$:

  $$\begin{array}{lll} \text{root} & (2) & \\ & (k) & \rightsquigarrow \quad (2)(3)\dots(k)(k+1) \end{array}$$

- Schröder rules for $(m,p) = (4,1)$:

  $$\begin{array}{llll} \text{root} & (2) & & \\ \text{rules} & (2) & \rightsquigarrow & (3)(3) \\ & (k) & \rightsquigarrow & (3)(4)\dots(k)(k+1)(k+1) \quad \text{if } k \geq 3 \end{array}$$

  See Figure 1 for the generating tree induced by these rules.

- Grand Dyck rules for $(m,p) = (4,2)$:

  $$\begin{array}{llll} \text{root} & (2) & & \\ \text{rules} & (2) & \rightsquigarrow & (3)(3) \\ & (k) & \rightsquigarrow & (3)(3)(4)\dots(k)(k+1) \quad \text{if } k \geq 3 \end{array}$$

- and for $(m,p) = (5,2)$:

  $$\begin{array}{llll} \text{root} & (2) & & \\ \text{rules} & (2) & \rightsquigarrow & (3)(3) \\ & (3) & \rightsquigarrow & (4)(4)(4) \\ & (k) & \rightsquigarrow & (4)(4)(5)\dots(k)(k+1)(k+1) \quad \text{if } k \geq 4. \end{array}$$

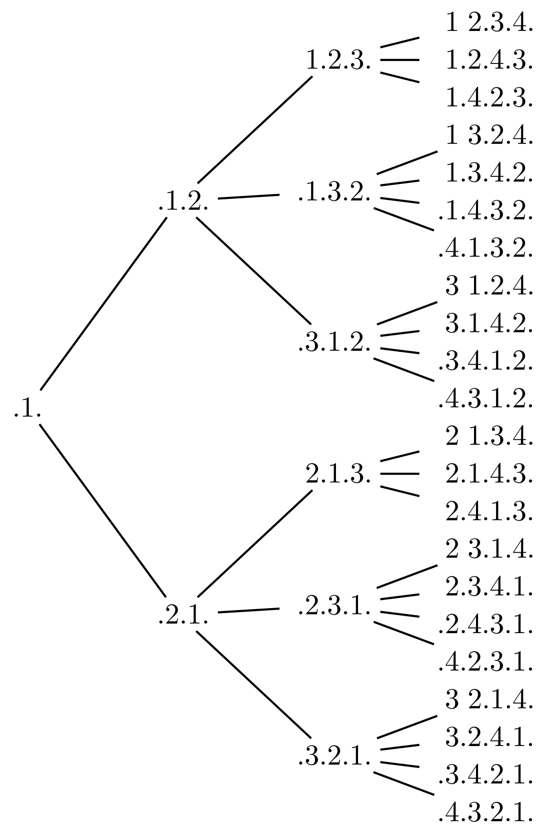

Figure 1: The first levels of the generating tree induced by the Schröder rules corresponding to $(m, p) = (4, 1)$. Active sites are represented by dots.

3 Production matrices

Any succession rule of the form given in (2) can be expressed as a root (labeled by ℓ_1 in this context) and a set of productions

$$\{(\ell_u) \rightsquigarrow (\ell_1)^{v(u,1)}(\ell_2)^{v(u,2)}(\ell_3)^{v(u,3)}\dots\}_{u\geq 1} \tag{3}$$

where $\{\ell_1, \ell_2, \dots\}$ is the set of admissible labels and for each u the ultimately zero integer sequence $\{v(u,k)\}_{k\geq 1}$ gives the repetition order.

The matrix

$$R = [v(i,j)]_{i,j\geq 1}$$

defined in [3] is called the *production matrix* of the succession rule (3). For example, the production matrix of the Dyck rule is

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 1 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

and that of Grand Dyck rule is

$$\begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 & \dots \\ 0 & 2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 1 & 1 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

The integer sequence corresponding to a succession rule (or equivalently, to a production matrix) is the sequence giving, for each n, the number of objects of size n produced by the succession rule. Observe that, the objects of size n are exactly those at level $n-1$ in the generating tree, considering the root at level zero.

For a production matrix P we denote by f_P the generating function of the integer sequence associated with P. Let denote by $u^\top$ the row vector $(1\,0\,0\,\dots 0)$, and by e the column vector $(1\,1\,1\,\dots 1)^\top$.

Theorem 3 (Theorem 3.2 of [3]). *Let a, b, c be three nonnegative integers, P and Q two production matrices related by*

$$P = \begin{bmatrix} b & a \cdot u^\top \\ c \cdot e & Q \end{bmatrix}.$$

Then the associated generating functions are related by

$$f_P(x) = \frac{1 + axf_Q(x)}{1 - bx - acx^2 f_Q(x)}.$$

In particular, this theorem gives the following corollary.

Corollary 1 (Corollary 3.1 of [3]). *Let a, b, c be three positive integers and P be an infinite production matrix of the form*

$$P = \begin{bmatrix} b & a \cdot u^\top \\ c \cdot e & P \end{bmatrix}.$$

Then $f_P(x)$ satisfies the quadratic equation

$$acx^2 f_P(x)^2 - (1 - bx - ax) f_P(x) + 1 = 0.$$

As a particular case ($b = c = 1$) of this corollary we obtain the following.

Corollary 2. *Let a be an integer and R a production matrix of the form*

$$R = \begin{bmatrix} 1 & a & 0 & 0 & 0 & \dots \\ 1 & 1 & a & 0 & 0 & \dots \\ 1 & 1 & 1 & a & 0 & \dots \\ 1 & 1 & 1 & 1 & a & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then

$$f_R(x) = \frac{N_a(x)}{2ax^2} \tag{4}$$

where

$$N_a(x) = 1 - (a+1)x - \sqrt{1 + (a-1)^2 x^2 - 2(a+1)x}.$$

Proof. Applying Corollary 1, we obtain the following functional equation for $f_R(x)$:

$$ax^2 f_R(x)^2 - (1 - x - ax) f_R(x) + 1 = 0.$$

Solving this equation leads to the desired expression. □

Lemma 3. *Let P be a production matrix of the form*

$$P = \begin{bmatrix} b & a & 0 & 0 & 0 & \dots \\ b & 1 & a & 0 & 0 & \dots \\ b & 1 & 1 & a & 0 & \dots \\ b & 1 & 1 & 1 & a & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then

$$f_P(x) = \frac{2x + N_a(x)}{x(2 - 2bx - bN_a(x))}, \tag{5}$$

where $N_a(x)$ is the same of the Corollary 2.

Proof. Applying Theorem 3, we obtain the following expression for $f_P(x)$:

$$f_P(x) = \frac{1 + axf_R(x)}{1 - bx - abx^2 f_R(x)},$$

where f_R is the generating function found in Corollary 2. Simplifying this expression leads to the desired formula. □

Lemma 4. *Let P and Q be two production matrices, and*

$$P = \begin{bmatrix} 0 & 2 & 0 & \dots & 0 & 0 & 0 & \dots \\ 0 & 0 & 3 & \dots & 0 & 0 & 0 & \dots \\ & & & \ddots & & & & \\ 0 & 0 & 0 & \dots & m-4 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots & 0 & m-3 & 0 & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & m-2 & \dots \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & & Q \end{bmatrix}.$$

Then the generating function of the numerical sequence associated with P is

$$f_P(x) = \sum_{i=0}^{m-4} (i+1)! \cdot x^i + (m-2)! \cdot x^{m-3} \cdot f_Q(x).$$

Proof. Theorem 3 gives as particular case ($b = c = 0$): if a is a nonnegative integer, P and M two production matrices with

$$P = \begin{bmatrix} 0 & a \cdot u^\top \\ 0 & M \end{bmatrix},$$

then

$$f_P(x) = 1 + a \cdot x f_M(x).$$

Now the statement holds by deleting the first row and column in P and iteratively applying the above relation. □

Theorem 4. *The generating function for the succession rule*

$$\begin{array}{lll} \text{root} & (2) & \\ \text{rules} & (k) & \rightsquigarrow \left\{ \begin{array}{ll} (k+1)^k & \text{if } k < m-1 \\ (m-1)^p(m)(m+1)\dots(k)(k+1)^{m-p-1} & \text{if } k \geq m-1 \end{array} \right. \end{array}$$

is given by

$$\Psi(x) = \sum_{i=0}^{m-4} (i+1)! \cdot x^i + (m-2)! \cdot x^{m-3} \cdot F(x),$$

where

$$F(x) = \frac{2x + N_{m-p-1}(x)}{x(2 - 2px - pN_{m-p-1}(x))}.$$

and

$$N_a(x) = 1 - (a+1)x - \sqrt{1 + (a-1)^2 x^2 - 2(a+1)x}.$$

Proof. The production matrix of the succession rule in Theorem 2 is

$$A_{m,p} = \begin{bmatrix} 0 & 2 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 3 & 0 & \dots & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 4 & \dots & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & 0 & \dots & m-2 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots & p & m-p-1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots & p & 1 & m-p-1 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots & p & 1 & 1 & m-p-1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

To determine an expression for the generating function of the numerical sequence associated with $A_{m,p}$ we apply Lemma 4 with

$$Q = \begin{bmatrix} p & m-p-1 & 0 & 0 & 0 & \dots \\ p & 1 & m-p-1 & 0 & 0 & \dots \\ p & 1 & 1 & m-p-1 & 0 & \dots \\ p & 1 & 1 & 1 & m-p-1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

By Lemma 3, with $a = m - p - 1$ and $b = p$, the generating function $F(x)$ of the production matrix Q is

$$F(x) = \frac{2x + N_{m-p-1}(x)}{x(2 - 2px - pN_{m-p-1}(x))}$$

and the result immediately follows applying Lemma 4.

□

Since in all of the generating trees considered above the root (the length one permutation) was considered to be at level zero, we have the following

Corollary 3. *The generating function of the sequence* $\{\mathrm{card}(\mathfrak{S}_n(\Sigma_m^p))\}_{n\geq 0}$ *is* $x \cdot \Psi(x)$.

Corollary 4. $\mathrm{card}(\mathfrak{S}_n(\Sigma_{p+1}^p)) = \begin{cases} n! & \text{if } n < p-1 \\ (p-1)! \cdot p^{n-p+1} & \text{otherwise.} \end{cases}$

We end this section with an open problem. For $1 \leq p < m$ define $\Gamma_m^p \subset \mathfrak{S}_m$ by

$$\Gamma_m^p = \{\pi \in \mathfrak{S}_m \mid m = \pi(m) \text{ and } m-1 \in \{\pi(m-p), \pi(m-p+1), \dots, \pi(m-1)\}\}.$$

Clearly, $\mathrm{card}(\Gamma_m^p) = \mathrm{card}(\Sigma_m^p)$ and Γ_m^p is still another generalisation of Γ_m, the set of patterns considered in [2] and defined in the beginning of the present paper. There is no trivial bijection between $\mathfrak{S}_n(\Sigma_m^p)$ and $\mathfrak{S}_n(\Gamma_m^p)$ and we have verified by computer, for several values of n, m and p, and think that the following is true.

Conjecture 1. *For any m and p, $1 \leq p < m$, Σ_m^p and Γ_m^p are Wilf equivalent, that is,* $\mathrm{card}(\mathfrak{S}_n(\Sigma_m^p)) = \mathrm{card}(\mathfrak{S}_n(\Gamma_m^p))$ *for* $n \geq 1$.

$m\backslash p$	1	2	3	4
2	1	–	–	–
3	Catalan	2^{n-1}	–	–
4	Schröder	$\binom{2n-2}{n-1}$	$2\cdot 3^{n-2}$ A025192	–
5	A054872			$6\cdot 4^{n-3}$ A084509

Table 1: Several instances of the sequence $\{\mathrm{card}(\mathfrak{S}_n(\Sigma_m^p))\}_{n\geq 0}$.

4 The sequences $\{\mathrm{card}(\mathfrak{S}_n(\Sigma_5^p))\}_{n\geq 0}$ for $1 \leq p \leq 4$

Here we give the first terms, the Sloane reference (if any) and the generating function corresponding to the sequences $\{\mathrm{card}(\mathfrak{S}_n(\Sigma_5^p))\}_{n\geq 0}$ for $1 \leq p \leq 4$.

- $\mathrm{card}(\mathfrak{S}_n(\Sigma_5^1))$

  first values: 0, 1, 2, 6, 24, 114, 600, 3372, 19824, ...

  Sloane: A054872

  generating function: $x\cdot\left(2-2x-\sqrt{1-8x+4x^2}\right)$

- $\mathrm{card}(\mathfrak{S}_n(\Sigma_5^2))$

  first values: 0, 1, 2, 6, 24, 108, 516, 2556, 12972 ...

  generating function: $x\cdot\left(1+2x+3x\cdot\frac{1-x-\sqrt{1-6x+x^2}}{x+\sqrt{1-6x+x^2}}\right)$

- $\mathrm{card}(\mathfrak{S}_n(\Sigma_5^3))$

  first values: 0, 1, 2, 6, 24, 102, 444, 1956, ...

  generating function: $x\cdot\left(1+2x+6x\cdot\frac{1-\sqrt{1-4x}}{-1+3\sqrt{1-4x}}\right)$

- $\mathrm{card}(\mathfrak{S}_n(\Sigma_5^4))$

  first values: 0, 1, 2, 6, 24, 96, 384, 1536, 6144, ...

  Sloane: A084509

  generating function: $x\cdot\frac{1-2x-2x^2}{1-4x}$

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