RESTRICTED 123-AVOIDING BAXTER PERMUTATIONS AND THE PADOVAN NUMBERS

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Abstract

Baxter studied a particular class of permutations by considering fixed points of the composite of commuting functions. This class is called Baxter permutations. In this paper we investigate the number of 123-avoiding Baxter permutations of length n that also avoid (or contain a prescribed number of occurrences of) another certain pattern of length k. In several interesting cases the generating function depends only on k and is expressed via the generating function for the Padovan numbers.

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1. Introduction

Pattern avoidance. Let $\mathfrak{S}_n$ denote the set of permutations of $\{1, 2, \ldots, n\}$, written in one-line notation, and suppose $\alpha \in \mathfrak{S}_n$ and $\tau \in \mathfrak{S}_k$ be two permutations. We say that α *contains* τ if there exists a subsequence $1 \le i_1 < i_2 < \cdots < i_k \le n$ such that $(\alpha_{i_1}, \ldots, \alpha_{i_k})$ is order-isomorphic to τ; in such a context τ is usually called a *pattern*. We say that α *avoids* τ, or is τ*-avoiding*, if such a subsequence does not exist. The set of all τ-avoiding permutations in $\mathfrak{S}_n$ is denoted by $\mathfrak{S}_n(\tau)$. For an arbitrary finite collection of patterns T, we say that α avoids T if α avoids all $\tau \in T$; the corresponding subset of $\mathfrak{S}_n$ is denoted by $\mathfrak{S}_n(T)$.

Generating trees. A *colored integer* is an integer or an integer with a subscript which is called color. For a colored integer e, $|e|$ denotes the value of e regardless its color and $|e| = e$ if e is simply an integer. For instance $|2_4| = 2$ and $|3| = 3$.

A *succession rule* on a set of colored integers Σ is a formal system consisting of a root $e_0 \in \Sigma$ and a set of *productions* of the form

$$\{(k) \rightsquigarrow (e_1(k))(e_2(k)) \cdots (e_{|k|}(k))\}_{k \in \Sigma}$$

with each $e_i(k) \in \Sigma$, $1 \le i \le |k|$, which explain how to derive, for any given *label* $k \in \Sigma$, its $|k|$ successors, $(e_1(k)), (e_2(k)), \ldots, (e_{|k|}(k))$. In this context Σ is called the set of labels.

A *generating tree* induced by a succession rule is an infinite tree with the root (at level zero) labeled by (e_0). Each node labeled by (k) has $|k|$ successors with the labels given by the production rules. A tree is a generating tree for a class of combinatorial objects if there exists a bijection between the objects of size n and the nodes at level $n-1$ in the tree. Notice that a class of combinatorial objects may have several generating trees. If a class of combinatorial objects has a generating tree induced by a succession rule on a finite set of labels then the generating function of the number of objects of a given size is a rational function $\frac{f}{g}$ (see for instance [11, pp. 242] and [12]). In addition if the degree of g equals the cardinality of the set of labels then this cardinality is minimal, i.e., there is no succession rule on a smaller set of labels inducing the same class of objects. In this case the succession rule is called minimal.

The *Padovan sequence* p_n [7] is given by $p_n = p_{n-2} + p_{n-3}$, $n \geq 3$, with the initial values $p_0 = 1$ and $p_1 = p_2 = 0$. The first terms of this sequence are

$$1, 0, 0, 1, 0, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, 151.$$

Baxter [1] studied a particular class of permutations by considering fixed points of the composite of commuting functions. This class is called *Baxter permutations*. A permutation $\pi \in \mathfrak{S}_n$ is called a *Baxter permutation* if it satisfies the two following conditions for all $1 \leq a < b < c < d \leq n$,

if $\pi_a + 1 = \pi_d$ and $\pi_b > \pi_d$ then $\pi_c > \pi_d$;

if $\pi_d + 1 = \pi_a$ and $\pi_c > \pi_a$ then $\pi_b > \pi_a$.

The Baxter permutations can be defined as the set of permutations in $\mathfrak{S}_n$ avoiding 2413 and 3142, the patterns being yet permitted when they are parts of 25314 and 41352 in the permutation, respectively; this class of pattern avoiding permutations is denoted by $\mathfrak{S}_n(25\bar{3}14, 41\bar{3}52)$, In [3] it is proved analytically that the number of Baxter permutations in $\mathfrak{S}_n$ is given by

$$\sum_{j=0}^{n-1} \frac{\binom{n+1}{j}\binom{n+1}{j+1}\binom{n+1}{j+2}}{\binom{n+1}{1}\binom{n+1}{2}}.$$

A bijective proof of this formula is given in [10]. Later, several papers enumerate number of Baxter permutations that satisfy certain set of conditions, as follows. We say that $\pi = \pi_1\pi_2\cdots\pi_n$ is an *alternating permutation* if it satisfies $\pi_1 < \pi_2 > \pi_3 < \pi_4 > \cdots$. A permutation π is said to be *double alternating permutation* if π and π^{-1} are alternating permutations. In [4] (see also references therein) it is proved that the number of alternating Baxter permutations of length $2n$ and $2n+1$ is given by C_n^2 and C_nC_{n+1}; respectively, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number. In [8] (see also references therein) is counted the number of double alternating Baxter permutations in $\mathfrak{S}_n$ and proved that this number is given by C_n.

In this paper we consider the case of 123-avoiding Baxter permutations that avoid other patterns, or the case of 123-avoiding Baxter permutations containing a given number of occurrences of another pattern of length k. In several interesting cases the generating function depends only on k and is expressed via Padovan numbers.

The paper is organized as follows. The case of Baxter permutations avoiding both 123 and one or two length k patterns is treated in Section 2. We describe a simple structure for the set of Baxter permutations avoiding 123. This structure gives a complete answer for several interesting cases, including the patterns $12\cdots k$ and $\tau = m(m-1)\cdots 1k(k-1)\cdots(m+1)$. The case of Baxter permutations avoiding 123 and containing τ exactly r times is treated in Section 3. The case of Baxter permutations avoiding the pattern 123 and containing a certain generalized pattern is treated in Section 4.

Most of the explicit solutions obtained in what follows involve generating trees and the generating function for the Padovan numbers.

2. Baxter permutations avoiding 123 and another pattern

Let $\mathfrak{B}_\tau(n)$ be the set of Baxter permutations in $\mathfrak{S}_n(123,\tau)$, with $\mathfrak{B}(n)=\mathfrak{B}_\varnothing(n)$, and $b_\tau(n)$ be its cardinality. We denote by $B_\tau(x)$ the corresponding generating function, that is, $B_\tau(x)=\sum_{n\geq 0} b_\tau(n)x^n$. The following proposition is the base for all the other results in this section.

Proposition 2.1. *Let $\pi\in\mathfrak{B}(n)$, then*

(a) *$\pi=(n,\pi')$ where $\pi'\in\mathfrak{B}(n-1)$;*
(b) *or there exist $1\leq j\leq i\leq n-1$ such that $\pi=(i,i-1,\dots,j,n,n-1,\dots,i+1,\pi')$, where $\pi'\in\mathfrak{B}(j-1)$.*

Proof. Let $\pi\in\mathfrak{B}(n)$ be a Baxter permutation such that $\pi_1=i$. If $i=n$ then the proposition holds immediately by definitions, so we assume that $1\leq i\leq n-1$. Let $\pi_d=n$. Since π is a 123-avoiding permutation we get that π contains the subsequence $(n,n-1,\dots,i+1)$ and $\pi_1>\cdots>\pi_{d-1}$. On the other hand, π is a Baxter permutation, there π has the form $(i,i-1,\dots,i-(d-2),n,n-1,\dots,i+1,\pi')$ where π' is a Baxter permutation in $\mathfrak{B}(i+1-d)$, hence the second case holds. □

2.1. 123-avoiding Baxter permutations.

As an application for Proposition 2.1 we get the generating tree and the number of 123-avoiding Baxter permutations in $\mathfrak{S}_n$.

Proposition 2.2. *The generating tree for the set of Baxter permutations in $\mathfrak{B}(n)$ is given by*

$$\begin{array}{llcl} \text{root} & (2_1) & & \\ \text{rule} & (2_1) & \rightsquigarrow & (2_1)(3) \\ & (2_2) & \rightsquigarrow & (2_1)(2_2) \\ & (3) & \rightsquigarrow & (2_1)(2_2)(3) \end{array} \tag{2.1}$$

Proof. Let $\pi\in\mathfrak{B}(n)$. If $\pi_1=n$, then we label π by (2_1); if $\pi_2=n$, then we label π by (3); and by (2_2) otherwise. When a permutation $\pi\in\mathfrak{B}(n)$ is labeled by (2_1) it has two successors: $(n+1,\pi)$ and $(n,n+1,\pi_2,\dots,\pi_n)$; they are labeled by (2_1) and (3), respectively. When π is labeled by (3) it has the form $(i,n,n-1,\dots,i+1,\pi')$ and has three successors: $(n+1,\pi)$, $(i+1,i,n+1,n,\dots,i+2,\pi')$ and $(i,n+1,n,\dots,i+1,\pi')$; they are labeled by (2_1), (2_2) and (3), respectively. When π is labeled by (2_2) it has the form $(i,i-1,\dots,j,n,n-1,\dots,i+1,\pi')$, $i>j$, and has two successors: $(n+1,\pi)$ and $(i+1,i,\dots,j,n+1,n,\dots,i+2,\pi')$; they are labeled by (2_1) and (2_2), respectively. Moreover, the unique Baxter permutation in $\mathfrak{B}(1)$ is labeled by (2_1) and any Baxter permutation in $\mathfrak{B}(n)$, $n>1$, can be uniquely obtained from a Baxter permutation in $\mathfrak{B}(n-1)$ by one of the three transformations above. □

Theorem 2.3. *The generating function for the number of 123-avoiding Baxter permutations in $\mathfrak{B}(n)$ is given by*

$$B_\varnothing(x)=\frac{(1-x)^2}{1-3x+2x^2-x^3}.$$

In other words, the number of 123-avoiding Baxter permutations in $\mathfrak{B}(n)$ is given by p_{3n+3}, the $(3n+3)$th Padovan number.

Proof. **First proof**. By Proposition 2.1, we have two possibilities for an arbitrary Baxter permutation $\pi\in\mathfrak{B}(n)$. Let us write an equation for $b_\varnothing(n)$. The contribution of the first case is $b_\varnothing(n-1)$ and the contribution of the second case for all $1 \le j \le i \le n-1$ is $b_\varnothing(j-1)$. Therefore, for all $n \ge 1$,

$$b_\varnothing(n) = b_\varnothing(n-1) + \sum_{i=1}^{n-1}\sum_{j=1}^{i} b_\varnothing(j-1). \tag{2.2}$$

So, $b_\varnothing(n) - b_\varnothing(n-1) = b_\varnothing(n-1) + \sum_{j=0}^{n-3} b_\varnothing(j)$, which implies that

$$b_\varnothing(n) = 3b_\varnothing(n-1) - 2b_\varnothing(n-2) + b_\varnothing(n-3).$$

Besides, $b_\varnothing(0) = b_\varnothing(1) = 1$ and $b_\varnothing(2) = 2$, hence $b_\varnothing(x) = \frac{(1-x)^2}{1-3x+2x^2-x^3}$.

Now let us prove that $b_\varnothing(n) = p_{3n+3}$ for all n. By using the definition of Padovan sequence we get

$$\begin{aligned} p_{3n+3} = p_{3n+1} + p_{3n} &= p_{3n} + p_{3n-1} + p_{3n-2} = 2p_{3n} + p_{3n-1} - p_{3n-3} = \\ &= 3p_{3n} - p_{3n-3} - (p_{3n} - p_{3n-3}) + (p_{3n-1} - p_{3n-3}) = \\ &= 3p_{3n} - p_{3n-3} - p_{3n-2} + p_{3n-4} = 3p_{3n} - p_{3n-3} - p_{3n-5} = \\ &= 3p_{3n} - 2p_{3n-3} + (p_{3n-3} - p_{3n-5}) = 3p_{3n} - 2p_{3n-3} + p_{3n-6}, \end{aligned}$$

hence, by induction on n we get the desired result.

Second proof. The number of (2_1)-labeled nodes at level $n \ge 1$ in the generating tree induced by (2.1) (considering the root at level 0) equals the total number of nodes at level $n-1$, which in turn equals the number of Baxter permutations in $\mathfrak{B}(n)$. The transfer matrix of the succession rule (2.1) is (see for instance [5])

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

and the number of (2_1)-labeled nodes at level n, or equivalently, the number of length n Baxter permutations avoiding 123 has the generating function

$$B_\varnothing(x) = \frac{\det(I - xA : 1, 1)}{\det(I - xA)}$$

where $(A : i, j)$ denotes the matrix obtained by removing the ith row and jth column in A (see [11, pp. 242]), and by simple calculation the desired result holds. $\square$

2.2. **A pattern $\tau = \mathbf{m}\cdots\mathbf{1}\mathbf{k}\cdots(\mathbf{m}+\mathbf{1})$.** Denote the permutation $m\cdots 1k\cdots(m+1)$ by $\tau[m,k]$. Now, let us consider the case $\tau \ne \varnothing$. We start by the following example.

Example 2.4. *Proposition 2.1 for $\pi = 132$ yields $b_{132}(n) = 2b_{132}(n-1)$. Besides, $b_{132}(1) = 1$, hence $b_{132}(n) = 2^{n-1}$ for all $n \ge 1$.*

The case of varying k is more interesting. As an extension of Example 2.4 let us consider the case $\tau = \tau[m,k]$. The next theorem shows that the corresponding generating function does not depend on m.

Theorem 2.5. *For $k \ge 3$ and $1 \le m \le k-1$ the cardinality of the set of Baxter permutations in $\mathfrak{B}(n)$ avoiding $\tau[m,k]$ does not depend on m.*

Proof. Let $k \ge 3$ and $1 < m \le k-1$, and let $\tau^{(1)} = \tau[m-1,k]$ and $\tau^{(2)} = \tau[m,k]$. We construct a bijection $\pi \hookrightarrow \tilde\pi$ from the set of Baxter permutations in $\mathfrak{B}(n)$ into itself. The permutation $\tilde\pi$ is defined recursively from π by $\tilde\pi = (n, \tilde{\pi'})$ if $\pi = (n, \pi')$, where $\pi' = \pi_2\cdots\pi_n$. Otherwise π has the form $(i, i-1, \dots, j, n, n-1, \dots, i+1, \pi')$, where $\pi' = \pi_{n-j+2}\pi_{n-j+3}\cdots\pi_n$. If $(i, i-1, \dots, j, n, n-1, \dots, i+1)$

- contains both $\tau^{(1)}$ and $\tau^{(2)}$ or contains neither $\tau^{(1)}$ nor $\tau^{(2)}$, then $\tilde{\pi} = (i, i-1, \dots, j, n, n-1, \dots, i+1, \tilde{\pi'})$.
- contains $\tau^{(2)}$ but not $\tau^{(1)}$, then $\tilde{\pi} = (m+j-2, \dots, j, n, n-1, \dots, i+1, i, m+j-1, \tilde{\pi'})$. In this case $k-m = n-i$ and $\tilde{\pi}$ contains $\tau^{(1)}$ but not $\tau^{(2)}$.
- contains $\tau^{(1)}$ but not $\tau^{(2)}$, then $\tilde{\pi} = (n-k+m, \dots, i, i-1, \dots, j, n, n-1, \dots, n-k+m+1, \tilde{\pi'})$. In this case $m-1 = j-i+1$ and $\tilde{\pi}$ contains $\tau^{(2)}$ but not $\tau^{(1)}$.

Clearly, $\pi \hookrightarrow \tilde{\pi}$ is a bijection from $\mathfrak{B}(n)$ into itself and it transforms a permutation avoiding $\tau^{(2)}$ into one avoiding $\tau^{(1)}$ and vice versa. Its restrictions $\mathfrak{B}_{\tau^{(2)}}(n) \tilde{\to} \mathfrak{B}_{\tau^{(1)}}(n)$ and $\mathfrak{B}_{\tau^{(1)}}(n) \tilde{\to} \mathfrak{B}_{\tau^{(2)}}(n)$ are bijections, inverses of each other. □

The restriction to $\mathfrak{B}_{1k\cdots32}(n)$ of the generating tree induced by (2.1) is given by the next proposition.

Proposition 2.6. *The generating tree for the set of Baxter permutations in $\mathfrak{B}_{1k\cdots32}(n)$ is given by*

- *if $k = 3$*

$$\begin{array}{llcl} \text{root} & (2_1) & & \\ \text{rule} & (2_1) & \rightsquigarrow & (2_1)(2_2) \\ & (2_2) & \rightsquigarrow & (2_1)(2_2) \end{array} \tag{2.3}$$

- *if $k > 3$*

$$\begin{array}{llcl} \text{root} & (2_1) & & \\ \text{rule} & (2_1) & \rightsquigarrow & (2_1)(3_1) \\ & (2_2) & \rightsquigarrow & (2_1)(2_2) \\ & (3_p) & \rightsquigarrow & \begin{cases} (2_1)(2_2)(3_{p+1}) & \text{if} \quad p < k-3, \\ (2_1)(2_2)(2_2) & \text{if} \quad p = k-3. \end{cases} \end{array} \tag{2.4}$$

Proof. For $k = 3$, by Proposition 2.1, the Baxter permutations in $\mathfrak{B}_{132}(n)$ have the form (n, π') or $(n-1, \dots, j, n, \pi')$. We label the first ones by (2_1) and they have two successors: $(n+1, n, \pi')$ and $(n, n+1, \pi')$, which are labeled by (2_1) and (2_2), respectively. The permutations of the form $(n-1, \dots, j, n, \pi')$ are labeled by (2_2) and they have also two successors: $(n+1, \pi)$ and $(n, n-1, \dots, j, n+1, \pi')$, which again are labeled by (2_1) and (2_2), respectively.

For $k > 3$, (2.4) results from (2.1) by limiting to $k-2$ the length of the sequence $n, n-1, \dots, i+1$ in each Baxter permutation π in $\mathfrak{B}(n)$ with $\pi_1 \neq n$. □

Theorem 2.7. *Let $k \geq 3$ and $1 \leq m \leq k-1$. Then*

$$B_{\tau[m,k]}(x) = \frac{1-x}{1-2x-x^3-x^4-\cdots-x^{k-1}}.$$

Proof. **First proof**. Let $\tau = \tau[m,k]$; by Proposition 2.1 we have two possibilities for an arbitrary Baxter permutation $\pi \in \mathfrak{B}_\tau(n)$. Let us write an equation for $b_\tau(n)$. The contribution of the first case is $b_\tau(n-1)$. The contribution of the second case, if $\pi_1 = i \leq m-1$ is $b_\tau(0) + b_\tau(1) + \cdots + b_\tau(i-1)$, if $m \leq \pi_1 = i \leq n-(k-m)$ equals $b_\tau(i+1-m) + b_\tau(i+2-m) + \cdots + b_\tau(i-1)$, otherwise (that is, $n-(k-m)+1 \leq \pi_1 = i \leq n-1$) equals $b_\tau(0) + b_\tau(1) + \cdots + b_\tau(i-1)$. Therefore, for all $n \geq 1$,

$$b_\tau(n) = b_\tau(n-1) + \sum_{i=0}^{m-2} \sum_{j=0}^{i} b_\tau(j) + \sum_{i=m-1}^{n-(k-m)-1} \sum_{j=i-m+2}^{i} b_\tau(j) + \sum_{i=n-(k-m)}^{n-2} \sum_{j=0}^{i} b_\tau(j),$$

so $b_\tau(n) - b_\tau(n-1) = b_\tau(n-1) + \sum_{j=n-(k-m)-m+1}^{n-3} b_\tau(j)$, which implies that

$$b_\tau(n) = 2b_\tau(n-1) + \sum_{j=n-k+1}^{n-3} b_\tau(j).$$

Besides, $b_\tau(n) = p_{3n+3}$ for all $n \le k-1$ (see Theorem 2.3), hence we multiply by x^n and add over all $n \ge k$ to get the desired result.

Second proof. By Theorem 2.5 it is enough to prove the result for $B_{1k(k-1)\cdots 2}(x)$ and $k \ge 3$. When $k = 3$, the transfer matrix of the system (2.3) is

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

and so $B_{132}(x) = \frac{1-x}{1-2x}$. When $k > 3$, the transfer matrix of the system (2.4) is

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & 1 & 0 & \dots & 0 \\ 1 & 1 & 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & & \ddots & \\ 1 & 1 & 0 & 0 & 0 & \dots & 1 \\ 1 & 2 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{(k-1)\times(k-1)}$$

and using the same techniques as in the second proof of Theorem 2.3 the result holds. $\square$

As a remark, if we fix m and let $k \to \infty$, then Theorem 2.7 yields that the number of Baxter permutations in $\mathfrak{B}(n)$ is given by p_{3n+3}, the $(3n+3)$th Padovan number (see Theorem 2.3).

Let us now consider simultaneous avoidance of two length k patterns of the form $m\cdots 21k\cdots(m+2)(m+1)$ and, as previously, we denote such a pattern by $\tau[m,k]$. We show that the corresponding generating function depends only on the length of patterns.

Theorem 2.8. *Let $k \ge 3$ and $1 < m \le k-1$. The number of 123-avoiding Baxter permutations in $\mathfrak{B}(n)$ avoiding both patterns $\tau^{(1)} = \tau[m-1,k]$ and $\tau^{(2)} = \tau[m,k]$ does not depend on m.*

Proof. Let $m \le k-2$ and $\tau^{(3)} = \tau[m+1,k]$. We construct a bijection $\pi \hookrightarrow \hat{\pi}$ from $\mathfrak{B}_{\tau^{(1)},\tau^{(2)}}(n)$ to $\mathfrak{B}_{\tau^{(2)},\tau^{(3)}}(n)$ defined by $\hat{\pi} = \pi$ if π does not contain $\tau^{(3)}$, and recursively as follows otherwise. If $\pi = (n, \pi')$ then $\hat{\pi} = (n, \hat{\pi'})$. Otherwise π has the form $(i, i-1, \dots, j, n, n-1, \dots, i+1, \pi')$. If $(i, i-1, \dots, j, n, n-1, \dots, i+1)$

- does not contain $\tau^{(3)}$, then $\hat{\pi} = (i, i-1, \dots, j, n, n-1, \dots, i+1, \hat{\pi'})$
- contains $\tau^{(3)}$, then $\hat{\pi} = (m-2+j, m-3+j, \dots, j, n, \dots, i+1, n-1+j, \hat{\pi'})$.

Thus, $\pi \hookrightarrow \hat{\pi}$ is invertible and so it is a bijection between $\mathfrak{B}_{\tau^{(1)},\tau^{(2)}}(n)$ and $\mathfrak{B}_{\tau^{(2)},\tau^{(3)}}(n)$. $\square$

The generating trees induced by the following succession rule are subtrees of that induced by (2.3) and (2.4), respectively, which in turn are subtrees of (2.1). Also, all of these succession rules are minimal.

Proposition 2.9. *The succession rules for the set of Baxter permutations in $\mathfrak{B}_{1k\cdots 32,21k\cdots 3}(n)$ are given by*

- *if* $k = 3$

$$\begin{array}{llll} \text{root} & (2) & & \\ \text{rule} & (2) & \rightsquigarrow & (2)(1) \\ & (1) & \rightsquigarrow & (2) \end{array} \tag{2.5}$$

- *if* $k > 3$

$$\begin{array}{llll} \text{root} & (2_1) & & \\ \text{rule} & (1) & \rightsquigarrow & (2_1) \\ & (2_1) & \rightsquigarrow & (2_1)(3_1) \\ & (2_2) & \rightsquigarrow & (2_1)(2_2) \\ & (3_p) & \rightsquigarrow & \begin{cases} (2_1)(2_2)(3_{p+1}) & \text{if} \quad p < k-3, \\ (2_1)(1)(2_2) & \text{if} \quad p = k-3. \end{cases} \end{array} \tag{2.6}$$

Proof. For $k = 3$, by Proposition 2.1, the Baxter permutations in $\mathfrak{B}_{132,213}(n)$ have the form (n, π') or $(n-1, n, \pi')$. The first ones are labeled by (2) and they have two successors: $(n+1, n, \pi')$ and $(n, n+1, \pi')$. The second ones are labeled by (1) and they have one successor: $(n+1, \pi)$.

The succession rule (2.1) can produce sequences $(i, i-1, \ldots, j, n, n-1, \ldots, i+1)$ of arbitrary length and it makes possible the creation of the patterns $1k\cdots 32$ and $21k\cdots 3$. So, for $k > 3$ the succession rule (2.6) results from (2.1) by imposing, in each π produced by (2.1) with $\pi_1 \neq n$, that the length of the sequence $(i, i-1, \ldots, j, n, n-1, \ldots, i+1)$ does not exceed $k-1$ whenever $n - i = k - 2$. $\square$

Theorem 2.10. *Let $k \geq 3$ and $1 < m \leq k-1$. The number of 123-avoiding Baxter permutations in $\mathfrak{B}(n)$ avoiding both patterns $\tau[m-1, k]$ and $\tau[m, k]$ is given by*

- *if* $k = 3$, $\frac{1}{1-x-x^2}$,
- *if* $k > 3$, $\frac{1-x}{1-2x-x^3-x^4-\cdots-x^{k-1}+x^k}$.

Proof. For $k = 3$ the the Baxter permutations in $\mathfrak{B}_{132,213}(n)$ are exactly the permutations in $\mathfrak{S}_n(123, 132, 213)$ and they are counted by the Fibonacci numbers, see [9].

For $k > 3$, by Theorem 2.8 it is enough to prove the result for $m = 2$. In this case the transfer matrix of the system (2.6) is

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & \ldots & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & & & & \ddots & \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & \ldots & 0 \end{bmatrix}_{k \times k}$$

and, again, by calculations the result holds. $\square$

2.3. **A pattern $\tau = (\mathbf{k-1})\mathbf{k}(\mathbf{k-2})\cdots\mathbf{21}$.** Again, by Proposition 2.1 it is easy to see that the number of Baxter permutations in $\mathfrak{B}_{231}(n)$ is given by $\binom{n}{2}+1$ for all $n \geq 0$. Indeed, $\mathfrak{B}_{231}(n)$ consists of the permutation $n(n-1)\cdots 1$ and $\binom{n}{2}$ permutations of the form $n(n-1)\cdots ji(i-1)\cdots 1(j-1)\cdots(i+1)$ with $1 \leq i < j \leq n$. The case of varying k is more interesting. As an extension of the above result let us consider the case of $\tau = (k-1)k(k-2)\cdots 21$.

Theorem 2.11. *Let $k \geq 3$; the number of Baxter permutations $b_{(k-1)k(k-2)\cdots 21}(n)$ is given by a polynomial of degree 2 with coefficients in $\mathcal{Q}$ for all $n \geq 2(k-3)$.*

Proof. Let $\tau = (k-1)k(k-2)\cdots 21$ and let us define $b_\tau(n; i_1, i_2, \ldots, i_m)$ to be the number of Baxter permutations $\pi \in \mathfrak{B}_\tau(n)$ such that $\pi_1\pi_2\cdots\pi_m = i_1 i_2 \cdots i_m$. In view of Proposition 2.1 it is easy to see that

$$b_\tau(n) = b_\tau(n; n) + \sum_{i=2}^{n-1} b_\tau(n; i, i-1) + \sum_{i=1}^{n-1} b_\tau(n; i, n).$$

By definitions we get $b_\tau(n; i, i-1) = b_\tau(n-1; i-1)$ and $b_\tau(n; n) = b_\tau(n-1)$, so by the fact that $\sum_{j=1}^{n} b_\tau(n; j) = b_\tau(n)$ we have

$$b_\tau(n) = 2b_\tau(n-1) - b_\tau(n-2) + \sum_{i=1}^{n-1} b_\tau(n; i, n).$$

On the other hand, in [6] it is proved that no sequence of length $d \geq (p-1)(q-1)$ avoids both $12\cdots p$ and $q(q-1)\cdots 1$. So, by using Proposition 2.1 we get that $b_\tau(n; i, n) = 0$ for all $i - 1 \geq 2(k-3)$ since the sequence $\pi_{n-i+2}, \pi_{n-i+3}, \ldots, \pi_n$ contains at least 123 or $(k-2)(k-3)\cdots 1$. Therefore, there exists a constant c such that

$$b_\tau(n) = 2b_\tau(n-1) - b_\tau(n-2) + c,$$

for all $n \geq 2(k-3)$. Thus, if $p = b_\tau(2(k-3)-2)$ and $q = b_\tau(2(k-3)-1)$, then by induction on n we can state that

$$b_\tau(2(k-3)+n) = q(n+2) - p(n+1) + c\binom{n+2}{2},$$

for all $n \geq 0$, as required. □

As an application of Theorem 2.11 and using the initial values of the sequence $b_{(k-1)k(k-2)\cdots 21}(n)$ we get the following.

Corollary 2.12.

(1) *For all $n \geq 0$, $b_{231}(n) = \frac{n}{2}(n-1) + 1$;*
(2) *For all $n \geq 2$, $b_{3421}(n) = \frac{3n}{2}(n-3) + 5$;*
(3) *For all $n \geq 4$, $b_{45321}(n) = 5n(n-6) + 52$;*
(4) *For all $n \geq 6$, $b_{564321}(n) = \frac{n}{2}(35n - 321) + 397$.*

We note that for all τ as in the previous theorem, possibly except for finitely many of them, there is no minimal succession rule for the set of Baxter permutation in $\mathfrak{B}(n)$ avoiding τ. Indeed, the generating function of the set under consideration is rational with the denominator a polynomial of degree two, and there exist finitely many succession rules on a set of two labels.

2.4. **Other statistics.** Another application for Proposition 2.1 is to consider statistics on 123-avoiding Baxter permutations according to the number of rises (number of rises for a permutation π is equal to $|\{i|\pi_i < \pi_{i+1}\}|$) or left-to-right maxima (number of left-to-right maxima for a permutation π is equal to $|\{i|\pi_i > \pi_j \text{ for all } j < i\}|$).

Theorem 2.13. *The number of Baxter permutations in $\mathfrak{B}(n)$ having r rises is $\binom{n+r}{3r}$. In other words, the generating function for the number of Baxter permutations in $\mathfrak{B}(n)$ having r rises is given by $\frac{x^{2r}}{(1-x)^{3r+1}}$.*

Proof. By Proposition 2.1 we get that for each Baxter permutations in $\mathfrak{B}(n)$ with r rises corresponds an integer sequence $\{x_i\}_{i=0}^{3r+1}$ with $x_{3j} < x_{3j+1} \le x_{3j+2} < x_{3j+3}$ for all j, $0 \le j \le r-1$, and $x_0 = 0$, $x_{3r} \le x_{3r+1} = n$. π is bijectively related to $\{x_i\}$ by

- $\pi_i = n+1-i$ if $x_{3j+1} \le i \le x_{3j+2}$
- $\pi_i < n+1-i$ if $x_{3j+2} < i \le x_{3j+3}$
- $\pi_i > n+1-i$ if $x_{3j} < i < x_{3j+1}$

and there are exactly $\binom{n+r}{3r}$ such sequences $\{x_i\}$. □

A 123-avoiding Baxter permutation in $\mathfrak{B}(n)$ has either one or two left-to-right maxima.

Theorem 2.14. *The number of* 123*-avoiding Baxter permutations in* $\mathfrak{B}(n)$ *having one left-to-right maxima is given by* p_{3n} *for all* $n \ge 1$*, and the number of* 123*-avoiding Baxter permutations in* $\mathfrak{B}(n)$ *having two left-to-right maxima is given by* $p_{3n+3} - p_{3n}$ *for all* $n \ge 2$*, where* p_m *is the mth Padovan number.*

Proof. If $\pi \in \mathfrak{B}(n)$ is a 123-avoiding Baxter permutation with one left-to-right maxima then $\pi = (n, \pi')$ and the number of such permutations is $b_\varnothing(n-1)$.

If π has two left-to-right maxima then $\pi = (i, i-1, \cdots, j, n, n-1, \ldots, i+1, \pi')$. For each j, $2 \le j \le n$, there are $j-1$ sequences $(i, i-1, \ldots, j, n, n-1, \ldots, i+1)$ and $b_\varnothing(j-1)$ permutations π'. So, the number of permutations π is $\sum_{j=2}^{n}(j-1)b_\varnothing(j-1)$ and, as in relation (2.2), it equals $b_\varnothing(n) - b_\varnothing(n-1)$. □

3. 123-avoiding Baxter permutations containing another pattern

Let $b_{\tau;r}(n)$ be the number of Baxter permutations in $\mathfrak{B}(n)$ containing τ exactly r times. We denote by $B_{\tau;r}(x)$ the corresponding generating function, that is, $B_{\tau;r}(x) = \sum_{n\ge 0} b_{\tau;r}(n)x^n$. Using similar arguments as in the proof of Theorem 2.7, along with Proposition 2.1, we have

Lemma 3.1. *Let* $k \ge 3$*,* $r \ge 0$*, and let* $\tau = \tau[m,k]$ *where* $1 \le m \le k-1$*. For all* $n \ge 1$*,*

$$b_{\tau;r}(n) = b_{\tau;r}(n-1) + \sum_{i=1}^{n-1}\sum_{j=1}^{i} b_{\tau;r-\binom{i-j+1}{m}\binom{n-i}{k-m}}(j).$$

3.1. **The case $\mathbf{r = 1}$ and $\tau = \tau[\mathbf{m}, \mathbf{k}]$.** Lemma 3.1 yields for $r = 1$ that

$$\begin{aligned} b_{\tau;1}(n) &= b_{\tau;1}(n-1) + b_{\tau;0}(n-k) + \sum_{i=0}^{m-2}\sum_{j=0}^{i} b_{\tau;1}(j) + \\ &+ \sum_{i=m-1}^{n-(k-m)-1}\sum_{j=i-m+2}^{i} b_{\tau;1}(j) + \sum_{i=n-(k-m)}^{n-2}\sum_{j=0}^{i} b_{\tau;1}(j), \end{aligned}$$

so

$$b_{\tau;1}(n) - b_{\tau;1}(n-1) = b_{\tau;1}(n-1) + b_{\tau;0}(n-k) - b_{\tau;0}(n-k-1) + \sum_{j=n-k+1}^{n-3} b_{\tau;1}(j),$$

equivalently

$$b_{\tau,1}(n) = 2b_{\tau,1}(n-1) + b_{\tau;0}(n-k) - b_{\tau;0}(n-k-1) + \sum_{j=n-k+1}^{n-3} b_{\tau;1}(n).$$

Besides, $b_{\tau;1}(n) = 0$ for all $n \le k-1$, hence by using Theorem 2.7 we get

Theorem 3.2. *Let $k \geq 3$ and $1 \leq m \leq k-1$. Then*

$$B_{\tau[m,k];1}(x) = \frac{x^k(1-x)^2}{(1-2x-x^3-x^4-\cdots-x^{k-1})^2}.$$

3.2. **The pattern $\tau = 132$.** In the current subsection we consider the case $\tau = 132$ and $r \geq 0$. As an application for Lemma 3.1 we have the following result.

Theorem 3.3. *We have*

(*i*) $b_{132;0}(n) = 2^{n-1}$ *,for all* $n \geq 1$*;*
(*ii*) $b_{132;1}(n) = n \cdot 2^{n-5}$*, for all* $n \geq 4$*,*
(*iii*) $b_{132;2}(n) = (n^2+13n-20)2^{n-10}$*, for all* $n \geq 7$*.*

Proof. By Lemma 3.1 for $r = 0$, $k = 3$, and $m = 1$ we get $b_{132;0}(n) = b_{132;0}(n-1) + \sum_{j=0}^{n-2} b_{132;0}(j)$. Besides, $b_{132;0}(0) = 1$, hence (i) holds.

Again, by Lemma 3.1 for $r = 1$, $k = 3$, and $m = 1$ we get

$$b_{132;1}(n) = b_{132;1}(n-1) + b_{132;0}(n-3) + \sum_{j=0}^{n-2} b_{132;1}(j).$$

Besides, $b_{132;0} = 0$, hence the rest is easy to check.

Similarly as the first two cases, (iii) holds. □

Remark 3.4. *In addition, Lemma 3.1 can be used to derive other examples for the choice of r and τ but we sufficient by the above two examples.*

4. Baxter permutation avoiding 123 and generalized pattern without dashes

Generalized patterns are introduced in [2]; they can impose the requirement that two adjacent letters in a pattern must be adjacent in the permutation. We write a classical pattern with dashes between any two adjacent letters of the pattern, say 1342, as 1-3-4-2; and, for example, the generalized pattern 24-3-1 means that if this pattern occurs in the permutation π, then the letters in the permutation π corresponding to 2 and 4 are adjacent. For example, the permutation $\pi = 35421$ has only two occurrences of the pattern 23-1, namely the subsequences 352 and 351, whereas π has four occurrences of the pattern 2-3-1, namely the subsequences 352, 351, 342 and 341.

Here we count a generalized pattern of length k without dashes in 1-2-3-avoiding Baxter permutations in $\mathfrak{B}(n)$. Let us consider the case of a generalized pattern $\tau = \tau[m,k]$ (without dashes) of length k.

Theorem 4.1. *Let $k \geq 3$ and $1 \leq m \leq k-1$. The generating function for the number of* 1-2-3*-avoiding Baxter permutations in $\mathfrak{B}(n)$ containing the generalized pattern $\tau[m,k]$ without dashes exactly r times is given by*

$$\frac{x^{kr}}{(1-x)^{r-1}(1-2x-x^3-\cdots-x^{k-1})^{r+1}}.$$

Proof. For $1 < m \leq k-1$ the transformation $\pi \hookrightarrow \tilde{\pi}$ given in Theorem 2.5 maps bijectively an occurrence of $\tau[m,k]$ into one of $\tau[m-1,k]$ and so it is enough to prove the statement for the particular pattern $\tau = (k-1)\cdots 21k$. Using the same argument as in the proof of Theorem 2.7 we get

$$b_{\tau;r}(n) = b_{\tau;r}(n-1) + \sum_{i=1}^{k-2}\sum_{j=1}^{i-1} b_{\tau;r}(j-1) + \\ + \sum_{i=k-1}^{n-1}\left(\sum_{j=1}^{i-k+1} b_{\tau;r-1}(j-1) + \sum_{j=i-k}^{i-1} b_{\tau;r}(j-1)\right),$$

therefore,

$$b_{\tau;r}(n) = 3b_{\tau;r}(n-1) - 2b_{\tau;r}(n-2) + b_{\tau;r}(n-3) - b_{\tau;r}(n-k) + b_{\tau;r-1}(n-k).$$

Hence, if multiplying by x^n and summing over all $n \geq 1$ then we have

$$B_{\tau;r}(x) = \frac{x^k}{(1-x)(1-2x-x^3-x^4-\cdots-x^{k-1})}B_{\tau;r-1}(x),$$

for all $r \geq 1$, together with $B_{\tau;0}(x) = \frac{1-x}{1-2x-x^3-x^4-\cdots-x^{k-1}}$. □

As a consequence of Theorem 4.1 we have the following result.

Theorem 4.2. *The number of Baxter permutations in $\mathfrak{B}(n)$ containing the generalized pattern* 132 *(or* 213*) exactly r times is given by*

$$\sum_{i=0}^{n-3r} 2^{n-3r-i}\binom{i+r-2}{r-2}\binom{n-3r-i+r}{r}.$$

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