The equidistribution of some length-three vincular patterns on $S_n(132)$

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Abstract

In 2012 Bóna showed the rather surprising fact that the cumulative number of occurrences of the classical patterns 231 and 213 is the same on the set of permutations avoiding 132, even though the pattern based statistics 231 and 213 do not have the same distribution on this set. Here we show that if it is required for the symbols playing the role of 1 and 3 in the occurrences of 231 and 213 to be adjacent, then the obtained statistics are equidistributed on the set of 132-avoiding permutations. Actually, expressed in terms of vincular patterns, we prove bijectively the following more general results: the statistics based on the patterns $2\underline{31}$, $2\underline{13}$ and $\underline{21}3$, together with other statistics, have the same joint distribution on $S_n(132)$, and so do the patterns $\underline{23}1$ and $3\underline{12}$; and up to trivial transformations, these statistics are the only based on length-three proper (not classical nor consecutive) vincular patterns which are equidistributed on a set of permutations avoiding a classical length-three pattern.

1 Introduction

In [2] Barnabei, Bonetti and Silimbani showed the equidistribution of some length-three consecutive patterns involvement statistics on the set of permutations avoiding the classical pattern 312 (or equivalently, 132), and in [4] Bóna showed the surprising fact that the total number of occurrences of the patterns 231 and 213 is the same on the set of 132-avoiding permutations, despite the pattern based statistics 231 and 213 having different distribution on this set. In [6], Homberger, generalizing Bóna's result, gave the total number of occurrences of each classical length-three pattern on the set of 123-avoiding permutations, and showed that the total number of occurrences of the pattern 231 is the same in the set of 123- and 132-avoiding permutations, despite the pattern based statistic 231 having different distribution on these two sets.

Vincular patterns, introduced by Babson and Steingrímsson [1], are a generalization of the notion of pattern where, for example, some entries are required to occur consecutively, and in [8] Mansour considered permutations avoiding 132 and containing various length-three vincular patterns exactly 0 or 1 times. Motivated by these, Burnstein and Elizalde gave in [5], in a much more general context, the total number of occurrences of any vincular pattern of length three on 231-avoiding (or equivalently, 132-avoiding) permutations, and more recently, Baxter [3] gave algorithmic methods to efficiently compute several statistics over some pattern-avoiding permutations.

In this paper we show that, on the set of 132-avoiding permutations, the vincular pattern based statistics $2\underline{31}$, $2\underline{13}$ and $\underline{21}3$ are equidistributed, and so are $\underline{23}1$ and $3\underline{12}$; and numerical evidence shows that, up to trivial transformations, these patterns are the only length-three proper (not classical nor consecutive) vincular patterns equidistributed on a set of permutations avoiding a classical length-three pattern.

It is worth to mention that, on the set of unrestricted permutations, the statistics $\underline{23}1$ and $3\underline{12}$ are trivially equidistributed, and so are $2\underline{31}$ and $2\underline{13}$ (which is all but obvious on 132-avoiding permutations), and this last distribution is different from that of $\underline{21}3$.

More precisely, in this paper we show bijectively the equidistribution on 132-avoiding permutations of the tuples of statistics

- $(2\underline{31}, 2\underline{13}, \mathsf{rlmin}, \mathsf{rlmax})$ and $(2\underline{13}, 2\underline{31}, \mathsf{rlmax}, \mathsf{rlmin})$,
- $(2\underline{31}, \mathsf{des})$ and $(2\underline{13}, \mathsf{des})$,
- $(\underline{21}3, \mathsf{des}, 12\rfloor)$ and $(2\underline{13}, \mathsf{des}, 12\rfloor)$,
- $(\underline{23}1, 3\underline{12}, \mathsf{des})$ and $(3\underline{12}, \underline{23}1, \mathsf{des})$,

where $\mathsf{rlmax}, \mathsf{rlmin}$ and des are respectively, the number of right-to-left maxima, right-to-left minima and descents. The corresponding bijections (the last of them being straightforward) are presented in Section 3.

2 Notations and definitions

A *permutation* of length n is a bijection from the set $\{1, 2, \ldots, n\}$ to itself and we write permutations in *one-line notation*, that is, as words $\pi = \pi_1\pi_2\ldots\pi_n$, where π_i is the image of i under π. Let S_n denote the set of permutations of length n.

2.1 Permutation patterns

Let $\sigma \in S_k$ and $\pi = \pi_1\pi_2\ldots\pi_n \in S_n$, $k \leq n$, be two permutations. One says that σ occurs as a (classical) pattern in π if there is a sequence $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ is order-isomorphic to σ. For example, 231 occurs as a pattern in 13452, and the three occurrences of it are 342, 352 and 452.

Vincular patterns have been introduced in [1] and they were extensively studied since then (see Chapter 7 in [7] for a comprehensive description of results on these patterns). Vincular patterns generalize classical patterns and they are defined as follows:

- Any pair of two adjacent letters may now be underlined, which means that the corresponding letters in the permutation must be adjacent. For example, the pattern $2\underline{13}$ occurs in the permutation 425163 four times, namely, as the subsequences 425, 416, 216 and 516. Note that the subsequences 426 and 213 are *not* occurrences of the pattern because their last two letters are not adjacent in the permutation.
- If a pattern begins (resp., ends) with a hook then its occurrence is required to begin (resp., end) with the leftmost (resp., rightmost) letter in the permutation. For example, there are two occurrences of the pattern $\lfloor 2\underline{13}$ in the permutation 425163, which are the subsequences 425 and 416.

We denote by $S_n(\sigma)$ the set of permutations in S_n avoiding the pattern σ.

2.2 Statistics

A *statistic* on a set of permutations is simply a function from the set to $\mathbb{N}$. Classical examples of statistics on S_n are the descent number

$$\mathsf{des}\,\pi = \mathrm{card}\,\{i \;:\; 1 \le i < n, \pi_i > \pi_{i+1}\},$$

and the inversion number

$$\mathsf{inv}\,\pi = \mathrm{card}\,\{(i,j) \;:\; 1 \le i < j \le n, \pi_i > \pi_j\}.$$

For example $\mathsf{des}\,45312 = 2$ and $\mathsf{inv}\,45312 = 8$.

In a permutation $\pi = \pi_1\pi_2\ldots\pi_n$, π_i is a *right-to-left maximum* if $\pi_i > \pi_j$ for all $j > i$; and the number of right-to-left maxima of π is denoted by $\mathsf{rlmax}\,\pi$. Similarly, π_i is a *right-to-left minimum* if $\pi_i < \pi_j$ for all $j > i$; and the number of right-to-left minima of π is denoted by $\mathsf{rlmin}\,\pi$.

For a set of permutations P, two statistics ST and ST' have the same distribution (or are equidistributed) on P if, for any k,

$$\mathrm{card}\{\pi \in P : \mathsf{ST}\,\pi = k\} = \mathrm{card}\{\pi \in P : \mathsf{ST}'\,\pi = k\},$$

and the tuples of statistics, or multistatistics, $(\mathsf{ST}_1, \mathsf{ST}_2, \ldots, \mathsf{ST}_p)$ and $(\mathsf{ST}'_1, \mathsf{ST}'_2, \ldots, \mathsf{ST}'_p)$ have the same distribution if, for any p-tuple $k = (k_1, k_2, \ldots, k_p)$,

$$\mathrm{card}\{\pi \in P : (\mathsf{ST}_1, \mathsf{ST}_2, \ldots, \mathsf{ST}_p)\,\pi = k\} = \mathrm{card}\{\pi \in P : (\mathsf{ST}'_1, \mathsf{ST}'_2, \ldots, \mathsf{ST}'_p)\,\pi = k\}.$$

For a permutation π and a (vincular) pattern σ we denote by $(\sigma)\,\pi$ the number of occurrences of this pattern in π, and (σ) becomes a permutation statistic. For example, $(\underline{21})\,\pi$ is $\mathsf{des}\,\pi$; $(21)\,\pi$ is $\mathsf{inv}\,\pi$; and $(12\rfloor)\pi$ is the last value of π minus one. Similarly, for a set of (vincular) patterns $\{\sigma, \tau, \ldots\}$, we denote by $(\sigma + \tau + \cdots)\,\pi$ the number of occurrences of these patterns in π.

2.3 Sum decomposition

For a permutation π, $|\pi|$ denotes its length (and so, $\pi \in S_{|\pi|}$), and for two permutations α and β, the *skew sum* of α and β, denoted $\alpha \ominus \beta$, is the permutation π of length $|\alpha| + |\beta|$ with

$$\pi_i = \begin{cases} \alpha_i + |\beta| & \text{if} \quad 1 \le i \le |\alpha|, \\ \beta_{i-|\alpha|} & \text{if} \quad |\alpha| + 1 \le i \le |\alpha| + |\beta|, \end{cases}$$

and the *direct sum* of α and β, denoted $\alpha \oplus \beta$, is the permutation π of length $|\alpha| + |\beta|$ with

$$\pi_i = \begin{cases} \alpha_i & \text{if} \quad 1 \le i \le |\alpha|, \\ \beta_{i-|\alpha|} + |\alpha| & \text{if} \quad |\alpha| + 1 \le i \le |\alpha| + |\beta|. \end{cases}$$

It is easy to check the following fact.

Fact 1. *For two permutations α and β, $\mathsf{des}\,\alpha \oplus \beta = \mathsf{des}\,\alpha + \mathsf{des}\,\beta$ and, when α and β are not empty, $\mathsf{des}\,\alpha \ominus \beta = \mathsf{des}\,\alpha + \mathsf{des}\,\beta + 1$.*

The next characterization of 132-avoiding permutations is folklore.

Fact 2. *For a non-empty permutation $\pi \in S_n$, $n \ge 1$, the following are equivalent:*

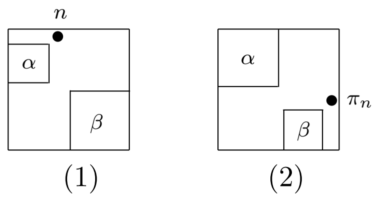

Figure 1: The two decompositions of $\pi \in S_n(132)$: (1) $\pi = (\alpha \oplus 1) \ominus \beta$, and (2) $\pi = \alpha \ominus (\beta \oplus 1)$.

- π *avoids* 132*,*
- π *can uniquely be written as* $(\alpha \oplus 1) \ominus \beta$*,*
- π *can uniquely be written as* $\alpha \ominus (\beta \oplus 1)$*,*

where α *and* β *are (possibly empty)* 132*-avoiding permutations. See Fig. 1.*

2.4 Permutation symmetries

For $\pi \in S_n$, the *reverse* and *complement* of π, denoted π^r and π^c respectively, are the permutations in S_n defined as:

- $\pi^r_i = \pi_{n-i+1}$,
- $\pi^c_i = n - \pi_i + 1$.

Both operations can naturally be extended to vincular patterns; for instance, the reverse of $\underline{23}1$ is $1\underline{32}$, and the complement of $\underline{23}1$ is $\underline{21}3$. These operations preserve pattern containment, in the sense that, if the (vincular) pattern σ is contained in the permutation π, then σ^r is contained in π^r, and σ^c is contained in π^c.

The *inverse* of π, denoted π^{-1}, is defined as:

- $\pi^{-1}_{\pi_i} = i$,

but, unlike the reverse and complement, it cannot be extended to vincular patterns: in general, the inverse of a vincular pattern is a *bivincular pattern* (see for example [7, p. 13] for its formal definition) a notion that we will not consider here.

3 The main results

3.1 Equidistribution of $(2\underline{31}, 2\underline{13}, \mathsf{rlmax}, \mathsf{rlmin})$ and $(2\underline{13}, 2\underline{31}, \mathsf{rlmin}, \mathsf{rlmax})$ on $S_n(132)$: bijection ϕ

We define a mapping ϕ on $S_n(132)$ and we will see that it is an involution, that is, a bijection from $S_n(132)$ into itself, which is its own inverse; and Theorem 1 below shows the desired equidistribution.

The mapping ϕ is recursively defined as follows: if π is the empty permutation (that is, $n = 0$), then $\phi(\pi) = \pi$; and if $\pi \in S_n(132)$, $n \geq 1$, with $\pi = \alpha \ominus (\beta \oplus 1)$ for some 132-avoiding permutations α and β, then

$$\phi(\pi) = \phi(\beta) \ominus (\phi(\alpha) \oplus 1).$$

See Fig. 2 for this definition.

Note that $\phi(\pi)$ is in some sense similar to π^{-1}, the inverse of π, but it is in fact different. Indeed, the inversion map when restricted to $S_n(132)$, satisfies: $(\alpha \ominus (\beta \oplus 1))^{-1} = (\beta^{-1} \oplus 1) \ominus \alpha^{-1}$.

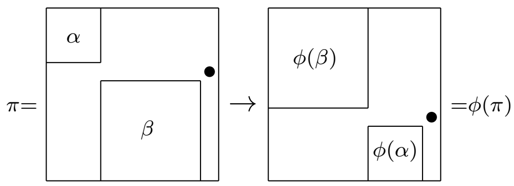

Figure 2: The recursive definition of $\phi(\pi)$.

By induction on n, from the definition of ϕ it follows that if $\pi \in S_n(132)$, then $\phi(\pi) \in S_n(132)$ and $\phi(\phi(\pi)) = \pi$, and in particular ϕ is a bijection of $S_n(132)$ to itself.

In the proof of the next theorem we will need the following result.

Proposition 1. *For any* $\pi \in S_n(132)$, *we have* $(12\rfloor)\,\pi = (21\rfloor)\,\phi(\pi)$, *and* $(21\rfloor)\,\pi = (12\rfloor)\,\phi(\pi)$.

Proof. If $\pi = \alpha \ominus (\beta \oplus 1)$ is a non-empty 132-avoiding permutation, then $(21\rfloor)\,\phi(\pi) = |\phi(\beta)| = |\beta| = (12\rfloor)\,\pi$; and $(12\rfloor)\,\phi(\pi) = |\phi(\alpha)| = |\alpha| = (21\rfloor)\,\pi$. □

Theorem 1. *If* $\pi \in S_n(132)$, *then*

$$(2\underline{13}, 2\underline{31}, \mathsf{rlmin}, \mathsf{rlmax})\,\phi(\pi) = (2\underline{31}, 2\underline{13}, \mathsf{rlmax}, \mathsf{rlmin})\,\pi.$$

Proof. By induction on n. Trivially, the statement holds for $n = 0$, and consider $\pi = \alpha \ominus (\beta \oplus 1) \in S_n(132)$, $n \geq 1$, for some 132-avoiding permutations α and β.

First we prove that $\mathsf{rlmax}\,\pi = \mathsf{rlmin}\,\phi(\pi)$. Indeed, by the induction hypothesis $\mathsf{rlmin}\,\phi(\alpha) = \mathsf{rlmax}\,\alpha$, and $\mathsf{rlmin}\,\phi(\pi) = 1 + \mathsf{rlmin}\,\phi(\alpha) = 1 + \mathsf{rlmax}\,\alpha = \mathsf{rlmax}\,\pi$. In addition, since ϕ is an involution, it follows that $\mathsf{rlmin}\,\pi = \mathsf{rlmax}\,\phi(\pi)$.

An occurrence of $2\underline{13}$ in $\phi(\pi)$ can be found either in $\phi(\alpha)$, or in $\phi(\beta)$, or has the form abc with ab an occurrence of $21\rfloor$ in $\phi(\alpha)$ and c the last symbol of $\phi(\pi)$. Thus $(2\underline{13})\,\phi(\pi) = (2\underline{13})\,\phi(\alpha) + (2\underline{13})\,\phi(\beta) + (21\rfloor)\,\phi(\alpha)$, and by the induction on n and Proposition 1 we have $(2\underline{13})\,\phi(\pi) = (2\underline{31})\,\alpha + (2\underline{31})\,\beta + (12\rfloor)\,\alpha$.

An occurrence of $2\underline{31}$ in π can be found either in α, or in β, or has the form abc with ab an occurrence of $12\rfloor$ in α and c the first symbol of β, if β is not empty, otherwise c is the last symbol of π. Thus $(2\underline{31})\,\pi = (2\underline{31})\,\alpha + (2\underline{31})\,\beta + (12\rfloor)\,\alpha$, and finally $(2\underline{13})\,\phi(\pi) = (2\underline{31})\,\pi$.

Moreover, since ϕ is an involution, it follows that $(2\underline{31})\,\phi(\pi) = (2\underline{13})\,\pi$, which completes the proof. □

3.2 Equidistribution of $(2\underline{31}, \mathsf{des})$ and $(2\underline{13}, \mathsf{des})$ on $S_n(132)$: bijection ψ

The 3-statistics $(2\underline{31}, 2\underline{13}, \mathsf{des})$ and $(2\underline{13}, 2\underline{31}, \mathsf{des})$ do not have the same distribution on $S_n(132)$; however in [5, Theorem 3.12] it is shown that $(2\underline{31})$ and $(2\underline{13})$ have the same distribution and Theorem 2 below refines this result by proving bijectively that $(2\underline{31})$ and $(2\underline{13})$, together with des, have the same joint distribution. In order to do this, we define the mapping

$\psi\colon S_n(132) \to S_n(132)$ by $\psi(1) = 1$ if $n = 1$, and for $n \geq 2$, $\psi(\pi)$ is defined recursively below, according to the following three cases: $\pi_n^{-1} = 1$, $2 \leq \pi_n^{-1} \leq n-1$, and $\pi_n^{-1} = n$; see Fig. 3.

Let $\pi \in S_n(132)$, $n \geq 2$.

1. If π has the form $1 \ominus \alpha$ (or equivalently, $\pi_n^{-1} = 1$), then $\psi(\pi)$ is simply $1 \ominus \psi(\alpha)$.

2. If π has the form $(\alpha \oplus 1) \ominus \beta$ for some non-empty permutations α and β (or equivalently, $2 \leq \pi_n^{-1} \leq n-1$), then $\psi(\pi)$ is obtained by considering the (possibly empty) permutations γ and δ defined by $\psi(\beta) = \gamma \ominus (\delta \oplus 1)$, and defining $\psi(\pi)$ as $((\psi(\alpha) \ominus (\delta \oplus 1)) \oplus 1) \ominus \gamma$. Note that α and β are 132-avoiding permutations, and by induction on n so are $\psi(\beta)$, γ and δ.

3. If π has the form $\alpha \oplus 1$ for some non-empty permutation α (or equivalently, $\pi_n^{-1} = n$), then $\psi(\pi)$ is obtained by considering the (possibly empty) permutations γ and δ with $\psi(\alpha) = (\gamma \oplus 1) \ominus \delta$, and defining $\psi(\pi)$ as $((\gamma \oplus 1) \oplus 1) \ominus \delta$. Note that, as above, α, γ and δ are 132-avoiding permutations.

From the above definition of ψ it is easy to check the following.

Proposition 2. *Let $\pi \in S_n(132)$, $n > 1$, and $\sigma = \psi(\pi)$.*

1. $\pi_n^{-1} = 1$ iff $\sigma_n^{-1} = 1$,

2. $2 \leq \pi_n^{-1} \leq n-1$ with $n > 2$ iff $\sigma_n^{-1} > 1$ and $\sigma_n^{-1} - \sigma_{n-1}^{-1} > 1$,

3. $\pi_n^{-1} = n$ iff $\sigma_n^{-1} > 1$ and $\sigma_n^{-1} - \sigma_{n-1}^{-1} = 1$.

Theorem 2. *The mapping ψ is a bijection on $S_n(132)$, and for any $\pi \in S_n(132)$, we have*

$$(2\underline{13}, \mathsf{des})\, \psi(\pi) = (2\underline{31}, \mathsf{des})\, \pi.$$

Proof. Let $\pi \in S_n(132)$. By induction on n, from Proposition 2 it follows that ψ is injective and thus bijective. In addition, from Fact 1 it follows that $\mathsf{des}\, \pi = \mathsf{des}\, \psi(\pi)$; see Fig. 3 where the number of descents is preserved for each intermediate permutation in the construction of $\psi(\pi)$ from π.

Now we show by induction on n that $(2\underline{13})\, \psi(\pi) = (2\underline{31})\, \pi$, for any $\pi \in S_n(132)$, $n \geq 1$. Clearly, for $n = 1$, $(2\underline{13})\, \psi(\pi) = (2\underline{31})\, \pi$, and let $\pi \in S_n(132)$, $n > 1$.

1. If $\pi_n^{-1} = 1$, then $\pi = 1 \ominus \alpha$ for some $\alpha \in S_{n-1}(132)$, and by definition, $\psi(\pi) = 1 \ominus \psi(\alpha)$. By the induction hypothesis we have $(2\underline{13})\, \psi(\alpha) = (2\underline{31})\, \alpha$, and thus $(2\underline{13})\, \psi(\pi) = (2\underline{13})\, \psi(\alpha) = (2\underline{31})\, \alpha = (2\underline{31})\, \pi$.

2. If $1 \leq \pi_n^{-1} \leq n-1$, let α, β, γ and δ be the permutations appearing in the second case of the definition of ψ (see Fig. 3), and we have

$$\begin{aligned}(2\underline{13})\, \psi(\pi) &= (2\underline{13})\, \psi(\alpha) + |\alpha| + (2\underline{13})\, \delta \oplus 1 + (2\underline{13})\, \gamma \\ &= (2\underline{13})\, \psi(\alpha) + |\alpha| + (2\underline{13})\, \psi(\beta),\end{aligned}$$

and by the induction hypothesis, $(2\underline{13})\, \psi(\pi) = (2\underline{31})\, \alpha + |\alpha| + (2\underline{31})\, \beta = (2\underline{31})\, \pi$.

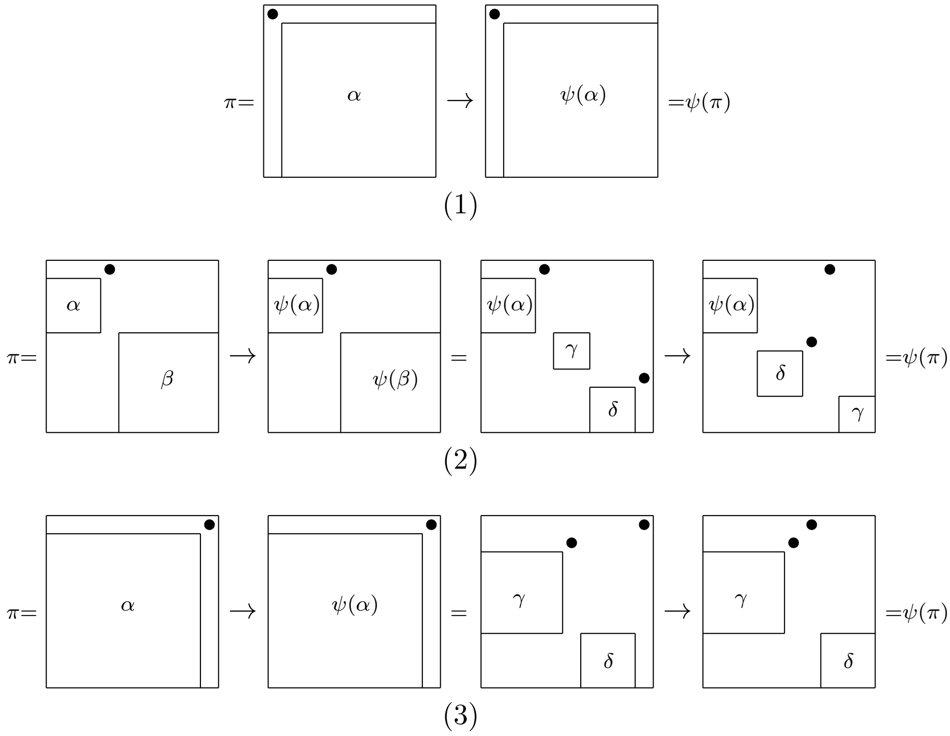

Figure 3: The three cases occurring in the definition of ψ: (1) $\pi_n^{-1} = 1$, (2) $2 \leq \pi_n^{-1} \leq n-1$, and (3) $\pi_n^{-1} = n$.

3. If $\pi_n^{-1} = n$, let α, γ and δ be the permutations appearing in the third case of the definition of ψ (see Fig. 3). Again, $(2\underline{13})\,\psi(\alpha) = (2\underline{31})\,\alpha$, and

$$\begin{array}{rcl}(2\underline{13})\,\psi(\pi) & = & (2\underline{13})\,\gamma \oplus 1 + (2\underline{13})\,\delta \\ & = & (2\underline{13})\,\psi(\alpha),\end{array}$$

and thus $(2\underline{13})\,\psi(\pi) = (2\underline{31})\,\alpha = (2\underline{31})\,\pi$. $\square$

3.3 Equidistribution of $(\underline{21}3, \mathsf{des}, 12\rfloor)$ and $(2\underline{13}, \mathsf{des}, 12\rfloor)$ on $S_n(132)$: bijection μ

Based on the previously defined bijection ψ we give a mapping μ on $S_n(132)$ and show that it is a bijection, and Theorem 3 proves the desired equidistribution.

Expressing in two different ways the major index of a permutation, Lemma 2 in [9] (see also Corollary 14 in [10]) shows that any (not necessarily 132-avoiding) permutation π satisfies $(2\underline{13} + 21\rfloor)\,\pi = (2\underline{31} + \underline{21})\,\pi$. Actually, $(\underline{21})\,\pi$ is equal to $\mathsf{des}\,\pi$, and the previous relation becomes

$$(2\underline{13} + 21\rfloor)\,\pi = (2\underline{31} + \mathsf{des})\,\pi. \tag{1}$$

Lemma 1. *For any permutation π, we have $(2\underline{13})\,\pi \oplus 1 = (2\underline{31} + \mathsf{des})\,\pi$.*

Proof. From relation (1) it follows that $(2\underline{13} + 21\rfloor)\,\pi \oplus 1 = (2\underline{31} + \mathsf{des})\,\pi \oplus 1$, and the statement holds by considering that $(21\rfloor)\,\pi \oplus 1 = 0$, $\mathsf{des}\,\pi \oplus 1 = \mathsf{des}\,\pi$, and $(2\underline{31})\,\pi \oplus 1 = (2\underline{31})\,\pi$. $\square$

In the proof of the next theorem we will use the following fact, which is easy to understand.

Fact 3. *For any permutation π, we have $(\underline{21}3)\,\pi \oplus 1 = (\underline{21}3 + \mathsf{des})\,\pi$.*

The mapping μ on $S_n(132)$ is recursively defined as follows: if π is the empty permutation, then $\mu(\pi) = \pi$; and if $\pi \in S_n(132)$, $n \geq 1$, with $\pi = \alpha \ominus (\beta \oplus 1)$ for some 132-avoiding permutations α and β, then

$$\mu(\pi) = \mu(\alpha) \ominus (\mu(\psi(\beta)) \oplus 1),$$

where ψ is the bijection define in Subsection 3.2. See Fig. 4 for this recursive construction.

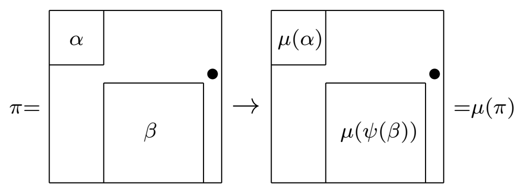

Figure 4: The recursive definition of $\mu(\pi)$.

Since ψ is a bijection on $S_n(132)$, $n \geq 0$, it follows that $\mu(\pi)$ avoids 132 whenever π does so, and thus $\mu(\pi) \in S_n(132)$ for any $\pi \in S_n(132)$. With the notations above, it is clear that $(12\rfloor)\,\mu(\pi) = (12\rfloor)\,\pi = |\beta|$ and considering again the bijectivity of ψ, by induction on n it follows that μ is injective, and thus bijective.

Theorem 3. *If $\pi \in S_n(132)$, then*

$$(\underline{21}3, \mathsf{des}, 12\rfloor)\,\mu(\pi) = (2\underline{13}, \mathsf{des}, 12\rfloor)\,\pi.$$

Proof. Clearly, $(12\rfloor)\,\mu(\pi) = (12\rfloor)\,\pi$, and the remaining of the proof is by induction on n. Obviously, the statement holds for $n = 0$, and consider $\pi = \alpha \ominus (\beta \oplus 1) \in S_n(132)$, $n > 0$, for some 132-avoiding permutations α and β.

The bijection μ preserves des statistic. Indeed, using the Iverson bracket notation, *i.e.* considering $[|\alpha| \neq 0]$ equal to 0 (resp. 1) if α is empty (resp. not empty), we have

$$\begin{aligned} \mathsf{des}\,\mu(\pi) &= \mathsf{des}\,\mu(\alpha) \ominus (\mu(\psi(\beta)) \oplus 1) \\ &= \mathsf{des}\,\mu(\alpha) + \mathsf{des}\,\mu(\psi(\beta)) + [|\alpha| \neq 0], \end{aligned}$$

and by the induction hypothesis it follows that $\mathsf{des}\,\mu(\pi) = \mathsf{des}\,\alpha + \mathsf{des}\,\psi(\beta) + [|\alpha| \neq 0]$, and since ψ preserves des we have $\mathsf{des}\,\mu(\pi) = \mathsf{des}\,\alpha + \mathsf{des}\,\beta + [|\alpha| \neq 0] = \mathsf{des}\,\alpha \ominus (\beta \oplus 1) = \mathsf{des}\,\pi$.

Finally, we show that $(\underline{21}3)\,\mu(\pi) = (2\underline{13})\,\pi$. Indeed, $(\underline{21}3)\,\mu(\pi) = (\underline{21}3)\,\mu(\alpha) \ominus (\mu(\psi(\beta)) \oplus 1) = (\underline{21}3)\,\mu(\alpha) + (\underline{21}3)\,\mu(\psi(\beta)) \oplus 1$, and

$$\begin{aligned} (\underline{21}3)\,\mu(\pi) &= (\underline{21}3)\,\mu(\alpha) + (\underline{21}3)\,\mu(\psi(\beta)) + \mathsf{des}\,\mu(\psi(\beta)) && \text{(by Fact 3)} \\ &= (2\underline{13})\,\alpha + (2\underline{13})\,\psi(\beta) + \mathsf{des}\,\beta && \text{(by the induction hypothesis} \\ & && \text{and } \mu \text{ and } \psi \text{ preserve } \mathsf{des}) \\ &= (2\underline{13})\,\alpha + (2\underline{31})\,\beta + \mathsf{des}\,\beta && \text{(by Theorem 2)} \\ &= (2\underline{13})\,\alpha + (2\underline{13})\,\beta \oplus 1, && \text{(by Lemma 1)} \end{aligned}$$

thus $(\underline{21}3)\,\mu(\pi) = (2\underline{13})\,\alpha \ominus (\beta \oplus 1) = (2\underline{13})\,\pi$.

□

3.4 Equidistribution of $(\underline{23}1, 3\underline{12}, \mathsf{des})$ and $(3\underline{12}, \underline{23}1, \mathsf{des})$

It is easy to see that the inverse of a permutation (defined at the end of Section 2) has the property that, if $\pi = \alpha \ominus (\beta \oplus 1)$, then $\pi^{-1} = (\beta^{-1} \oplus 1) \ominus \alpha^{-1}$ (see Fig. 5).

As mentioned before, the inverse of a vincular pattern is no longer a vincular pattern, however we have the following result.

Proposition 3. *If $\pi \in S_n(132)$, then $(\underline{23}1, 3\underline{12}, \mathsf{des})\,\pi^{-1} = (3\underline{12}, \underline{23}1, \mathsf{des})\,\pi$.*

Proof. Trivially, the statement holds for $n = 0$, and consider $\pi = \alpha \ominus (\beta \oplus 1) \in S_n(132)$, $n > 0$, for some 132-avoiding permutations α and β.

If α is empty, then $\mathsf{des}\,\pi^{-1} = \mathsf{des}\,\beta^{-1}$ and $\mathsf{des}\,\pi = \mathsf{des}\,\beta$; otherwise, $\mathsf{des}\,\pi^{-1} = \mathsf{des}\,\beta^{-1} + \mathsf{des}\,\alpha^{-1} + 1$ and $\mathsf{des}\,\pi = \mathsf{des}\,\alpha + \mathsf{des}\,\beta + 1$. In both cases, by induction on n it follows that $\mathsf{des}\,\pi^{-1} = \mathsf{des}\,\pi$.

An occurrence of $\underline{23}1$ in π^{-1} can be found either in β^{-1}, or in α^{-1}, or when α and β are not empty, has the form abc with ab two consecutive increasing symbols in $\beta^{-1} \oplus 1$ and c a symbol of α^{-1}. Similarly, an occurrence of $3\underline{12}$ in π can be found either in α, or in β, or when α and β are not empty, has the form abc with a a symbol of α and bc two consecutive increasing symbols in $\beta \oplus 1$. Thus, $(\underline{23}1)\,\pi^{-1} = (\underline{23}1)\,\beta^{-1} + (\underline{23}1)\,\alpha^{-1} + (\underline{12})(\beta^{-1} \oplus 1) \cdot |\alpha^{-1}|$ and $(3\underline{12})\,\pi = (3\underline{12})\,\alpha + (3\underline{12})\,\beta + |\alpha| \cdot (\underline{12})(\beta \oplus 1)$. But $(\beta \oplus 1)^{-1} = \beta^{-1} \oplus 1$ is a 132-avoiding permutation, and $(\underline{12})\gamma = (\underline{12})\gamma^{-1}$ for any 132-avoiding permutation. Finally, by induction on n it follows that $(\underline{23}1)\,\pi^{-1} = (3\underline{12})\,\pi$.

Since $\pi \mapsto \pi^{-1}$ is an involution on $S_n(132)$, it follows that $(3\underline{12})\,\pi^{-1} = (\underline{23}1)\,\pi$, and the statement holds. □

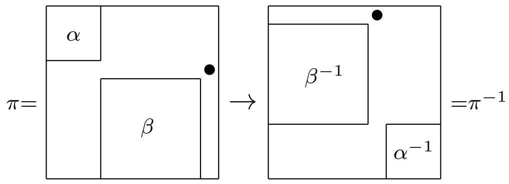

Figure 5: The recursive construction of π^{-1}, for $\pi \in S_n(132)$, $n \geq 1$.

4 Conclusions

We have shown bijectively the joint equidistribution on the set $S_n(132)$ of some length-three vincular patterns together with other statistics. In particular, for the sets of vincular patterns $\{2\underline{31}, 2\underline{13}, \underline{21}3\}$ and $\{\underline{23}1, 3\underline{12}\}$, we showed that the patterns within each set are equidistributed on $S_n(132)$. By applying permutation symmetries, other similar results can be derived. For instance, from the equidistribution of $\underline{21}3$ and $2\underline{13}$ on $S_n(132)$ (belonging to the first set, see Subsection 3.3) it follows, by applying

- the reverse operation, the equidistribution of $3\underline{12}$ and $\underline{31}2$ on $S_n(231)$,
- the complement operation, the equidistribution of $\underline{23}1$ and $2\underline{31}$ on $S_n(312)$, and
- the complement and the reverse operations (in any order), the equidistribution of $1\underline{32}$ and $\underline{13}2$ on $S_n(213)$.

Moreover, computer experiments show that, up to these two symmetries, the patterns in $\{2\underline{31}, 2\underline{13}, \underline{21}3\}$ and those in $\{\underline{23}1, 3\underline{12}\}$ are the only length-three proper (not classical nor consecutive) vincular patterns which are equidistributed on a set of permutations avoiding a classical length-three pattern.

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