Restricted compositions and permutations: from old to new Gray codes

V. Vajnovszki and R. Vernay
LE2I UMR CNRS 5158, Université de Bourgogne
B.P. 47 870, 21078 DIJON-Cedex France
e-mail: {vvajnov}{remi.vernay}@u-bourgogne.fr

March 30, 2011

Abstract

Any Gray code for a set of combinatorial objects defines a total order relation on this set: $\boldsymbol{x}$ is less than $\boldsymbol{y}$ if and only if $\boldsymbol{y}$ occurs after $\boldsymbol{x}$ in the Gray code list. Let $\prec$ denote the order relation induced by the classical Gray code for the product set (the natural extension of the Binary Reflected Gray Code to k-ary tuples). The restriction of $\prec$ to the set of compositions and bounded compositions gives known Gray codes for those sets. Here we show that $\prec$ restricted to the set of bounded *compositions of an interval* yields still a Gray code. An n-composition of an interval is an n-tuple of integers whose sum lies between two integers; and the set of bounded n-compositions of an interval simultaneously generalizes product set and compositions of an integer, and so $\prec$ put under a single roof all these Gray codes.

As a byproduct we obtain Gray codes for permutations with a number of inversions lying between two integers, and with even/odd number of inversions or cycles. Such particular classes of permutations are used to solve some computational difficult problems.

1 Introduction

Roughly speaking, a Gray code is a listing of the objects in a combinatorial family so that successive objects differ 'in some pre-specified small way' [5]. Here we adhere to the definition given in [14]: a Gray code for a combinatorial family is a listing of the objects in the family so that successive objects differ by a number of changes bounded independently of the object-size. Alternative more restrictive definitions for Gray codes exist in literature; they are obtained by imposing additional constraints. For instance, in [8, Section 7.2.1.3, page 10] is given an example of a Gray code for combinations in binary word representation where two consecutive words differ in two positions and all values between them are zeros. And in [12] is defined a Gray code for generalized Dyck words where two consecutive words differ in two positions which are either adjacent or separated by a zero.

For a given integer n, $\mathbb{N}^n$ denotes the set of all integer n-tuples and we adopt the convention that lower case bold letters represent such n-tuples; e.g., $\boldsymbol{w} = w_1 w_2 \dots w_n$.

An n-composition of an integer k is a tuple $\boldsymbol{c}$ with $c_1 + c_2 + \ldots + c_n = k$. Knuth gave[1] a definition of a Gray code for the set of n-compositions of an integer which is defined recursively by Wilf [15] and implemented iteratively by Klingsberg [7]. For two n-tuples $\boldsymbol{b}$ and $\boldsymbol{c}$, $\boldsymbol{c}$ is said

[1]unpublished answer to a question of Nijenhuis and Wilf.

$\boldsymbol{b}$*-bounded* if $0 \le c_i \le b_i$, for all i, $1 \le i \le n$. Walsh [13] gave a loopless algorithm for generating a Gray code for bounded compositions of an integer, and in particular for Knuth's Gray code.

A $\boldsymbol{b}$*-bounded n-composition of the integer interval* $[k, \ell]$ is a $\boldsymbol{b}$-bounded tuple $\boldsymbol{c}$ with $k \le c_1 + c_2 + \ldots + c_n \le \ell$. In particular, when $k = \ell$ we retrieve the notion of bounded n-composition of k, and when $k = \ell = b_1 = b_2 = \ldots = b_n$ that of classical composition; and when $k = 0$ and $\ell = b_1 + b_2 + \ldots + b_n$ we retrieve the product set $[b_1] \times [b_2] \times \ldots \times [b_n]$. So, bounded compositions of an interval simultaneously generalize product set and (bounded) compositions of an integer.

In this paper we re-express Walsh's and Knuth's Gray codes for (bounded) compositions of an integer in terms of a unique order relation, and so Walsh's Gray code becomes a sublist of Knuth's one, which in turn is a sublist of the *Reflected Gray Code*. Based on this order relation we generalize Knuth's and Walsh's Gray codes to bounded compositions of *an interval* $[k, \ell]$. We apply these results to obtain Gray codes for permutations with a number of inversions ranging between two integers, and with an even/odd number of inversions or cycles. Such particular classes of permutations are used to solve some computational difficult problems, see for instance [2].

2 Notations and definitions

For an integer $m \in \mathbb{N}$, $[m]$ denotes the set $\{0, 1, \ldots, m\}$ and for an n-tuple $\boldsymbol{b} \in \mathbb{N}^n$:

- $[\boldsymbol{b}]$ is the product set $[b_1] \times [b_2] \times \ldots \times [b_n]$, and
- $||\boldsymbol{b}||$ is the componentwise sum of $\boldsymbol{b}$, i.e., $||\boldsymbol{b}|| = \sum_{i=1}^{n} b_i$.

The Reflected Gray Code for the product set $[\boldsymbol{b}]$, denoted here by $\mathcal{G}_n(\boldsymbol{b})$, is the natural extension of the Binary Reflected Gray Code [4] to this set. $\mathcal{G}_n(\boldsymbol{b})$ was defined recursively by Er in [1] by the relation below and generated looplessly by Williamson in [16, p. 112].

$$\mathcal{G}_n(\boldsymbol{b}) = \begin{cases} \emptyset & \text{if} \quad n = 0, \\ 0\mathcal{G}_{n-1}(\boldsymbol{b}'),\ 1\overline{\mathcal{G}_{n-1}(\boldsymbol{b}')},\ 2\mathcal{G}_{n-1}(\boldsymbol{b}'),\ \ldots,\ b_1\mathcal{G}'_{n-1}(\boldsymbol{b}') & \text{if} \quad n > 0, \end{cases} \tag{1}$$

where $\boldsymbol{b}' = b_2 b_3 \cdots b_n$, $\overline{\mathcal{G}_{n-1}(\boldsymbol{b}')}$ is the reverse of $\mathcal{G}_{n-1}(\boldsymbol{b}')$ and $\mathcal{G}'_{n-1}(\boldsymbol{b}')$ is $\mathcal{G}_{n-1}(\boldsymbol{b}')$ or $\overline{\mathcal{G}_{n-1}(\boldsymbol{b}')}$ according as b_1 is even or odd. In $\mathcal{G}_n(\boldsymbol{b})$ two consecutive tuples differ in a single position and by $+1$ or -1 in this position, see the first column of the Table 1 for the list $\mathcal{G}_3(2\,1\,3)$.

The *Reflected Gray Code Order* $\prec$ on $\mathbb{N}^n$ is defined as: $\boldsymbol{x} = x_1 x_2 \ldots x_n$ is less than $\boldsymbol{y} = y_1 y_2 \ldots y_n$ in $\prec$ order, and we denote it by $\boldsymbol{x} \prec \boldsymbol{y}$, if either

- $\sum_{j=1}^{i-1} x_j$ is even and $x_i < y_i$, or
- $\sum_{j=1}^{i-1} x_j$ is odd and $x_i > y_i$,

where i is the leftmost position with $x_i \neq y_i$. It is easy to see that $\mathcal{G}_n(\boldsymbol{b})$ lists tuples in $[\boldsymbol{b}]$ in $\prec$ order.

In the following we introduce the notions of successor, and the first and last tuple in a set; unless explicitly specified otherwise, they are considered with respect to $\prec$ order:

- for $A \subset \mathbb{N}^n$, $\text{first}(A)$ and $\text{last}(A)$ stand for the first and last tuple (in $\prec$ order) in the set A,
- for $\boldsymbol{c} \in [\boldsymbol{b}]$, $\text{succ}^{\boldsymbol{b}}(\boldsymbol{c})$ is the successor of $\boldsymbol{c}$ (in $\prec$ order) in the product set $[\boldsymbol{b}]$,
- for a set A of tuples, $\mathcal{A}$ is the *ordered list* of tuples in A, listed in $\prec$ order.

3 Bounded compositions of an interval

Definition 1. For three integers $n \in \mathbb{N}$, $k \in \mathbb{Z}$ and $\ell \in \mathbb{N}$ with $k \leq \ell$, and an n-tuple $\boldsymbol{b} = b_1 b_2 \dots b_n \in \mathbb{N}^n$, a *$\boldsymbol{b}$-bounded n-composition of the interval* $[k, \ell]$ is a n-tuple $\boldsymbol{c} = c_1 c_2 \dots c_n \in \mathbb{N}^n$ such that

- $k \leq \sum_{i=1}^{n} c_i \leq \ell$, and
- $0 \leq c_i \leq b_i,$ for $1 \leq i \leq n$.

We denote by $W_{k,\ell}^{\boldsymbol{b}}$ the set of $\boldsymbol{b}$-bounded n-compositions of $[k, \ell]$, and obviously $W_{k,\ell}^{\boldsymbol{b}} = \cup_{i=k}^{\ell} W_{i,i}^{\boldsymbol{b}}$. For $\boldsymbol{c} \in W_{k,\ell}^{\boldsymbol{b}}$, $\mathrm{succ}_{k,\ell}^{\boldsymbol{b}}(\boldsymbol{c})$ denotes the successor of $\boldsymbol{c}$ (in $\prec$ order) in the set $W_{k,\ell}^{\boldsymbol{b}}$ and in the following we will omit the upper index $\boldsymbol{b}$ when it does not create ambiguity.

Remark that in the previous definition k can be a negative number; in this case $W_{k,\ell} = W_{0,\ell}$. Similarly, $W_{k,\ell} = W_{k,||\boldsymbol{b}||}$ if $\ell \geq ||\boldsymbol{b}||$.

Remark 1. As particular cases we obtain:

- $W_{0,||\boldsymbol{b}||}$ is the product set $[\boldsymbol{b}]$ generated looplessly in [16, p. 112],
- $W_{k,k}$ is the set of $\boldsymbol{b}$-bounded n-compositions of k generated looplessly in [13],
- If $\boldsymbol{b} = [k]^n$, then $W_{k,k}$ is the set of unrestricted n-compositions of k.

It can happen that $\mathrm{first}(W_{k,\ell}) = \mathrm{first}(W_{k+1,\ell+1})$. This case occurs only if $00\dots0$ belongs both to $W_{k,\ell}$ and $W_{k+1,\ell+1}$, and so when $k+1 \leq 0$. Similarly, it can happen that $\mathrm{last}(W_{k,\ell}) = \mathrm{last}(W_{k+1,\ell+1})$, and this case occurs only if $\mathrm{last}([\boldsymbol{b}])$ belongs both to $W_{k,\ell}$ and $W_{k+1,\ell+1}$. Formally, we have:

Lemma 1.

(i) $\mathrm{first}(W_{k+1,\ell+1})$ *and* $\mathrm{first}(W_{k,\ell})$ *are either both equal to* $00\dots0$*, or they differ in precisely one position and with difference 1 in this position.*

(ii) $\mathrm{last}(W_{k+1,\ell+1})$ *and* $\mathrm{last}(W_{k,\ell})$ *are either both equal to* $\mathrm{last}([\boldsymbol{b}])$*, or they differ in precisely one position and with difference 1 in this position.*

(iii) If $\boldsymbol{s} = \mathrm{last}(W_{k+1,\ell+1})$ *differs from* $\boldsymbol{t} = \mathrm{last}(W_{k,\ell})$ *in position* i*, then* $s_j = t_j \in \{0, b_j\}$ *for all* $j \neq i$*.*

(iv) $\mathrm{last}(W_{k,\ell}) \in \{\mathrm{last}(W_{k,k}), \mathrm{last}(W_{\ell,\ell}), \mathrm{last}([\boldsymbol{b}])\}$.

Proof. (i): $\mathrm{first}(W_{k,\ell})$ is the lexicographically least tuple $\boldsymbol{c} \in [\boldsymbol{b}]$ with $||\boldsymbol{c}|| = \max\{0, k\}$, and $\mathrm{first}(W_{k+1,\ell+1})$ the lexicographically least tuple $\boldsymbol{d} \in [\boldsymbol{b}]$ with $||\boldsymbol{d}|| = \max\{0, k+1\}$, and the result follows.
(ii): Suppose that $\mathrm{last}(W_{k,\ell}) \neq \mathrm{last}([\boldsymbol{b}])$ (that is, $\mathrm{last}([\boldsymbol{b}]) \notin W_{k,\ell}$). The case $\mathrm{last}(W_{k+1,\ell+1}) \neq \mathrm{last}([\boldsymbol{b}])$ (that is, $\mathrm{last}([\boldsymbol{b}]) \notin W_{k+1,\ell+1}$) is similar.

- If $0 \leq \ell < b_1$, then

$$\mathrm{last}(W_{k,\ell}) = \ell 0 \dots 0$$

and

$$\mathrm{last}(W_{k+1,\ell+1}) = (\ell+1) 0 \dots 0$$

and the statement follows.

• If $\ell \geq b_1$ and b_1 is odd, then

$$\text{last}(W_{k,\ell}) = b_1 \cdot \text{first}(W_{k-b_1,\ell-b_1})$$

and

$$\text{last}(W_{k+1,\ell+1}) = b_1 \cdot \text{first}(W^{\boldsymbol{b}'}_{k+1-b_1,\ell+1-b_1})$$

with $\boldsymbol{b}' = b_2 b_3 \dots b_n$. But $\text{first}(W^{\boldsymbol{b}'}_{k-b_1,\ell-b_1}) \neq 00\dots0$, otherwise $\text{last}(W_{k,\ell}) = \text{last}([\boldsymbol{b}])$, and by (i) of the present lemma the statement follows.

• If $k \geq b_1$ and b_1 is even, then

$$\text{last}(W_{k,\ell}) = b_1 \cdot \text{last}(W^{\boldsymbol{b}'}_{k-b_1,\ell-b_1})$$

and

$$\text{last}(W_{k+1,\ell+1}) = b_1 \cdot \text{last}(W^{\boldsymbol{b}'}_{k+1-b_1,\ell+1-b_1}).$$

But $\text{last}(W^{\boldsymbol{b}'}_{k-b_1,\ell-b_1}) \neq \text{last}([\boldsymbol{b}'])$, otherwise $\text{last}(W_{k,\ell}) = \text{last}([\boldsymbol{b}])$, and induction on n completes the proof.
(iii) and (iv) are consequences of the proof of (ii).

□

If $\boldsymbol{c} \in W_{k,\ell}$ and $\text{succ}(\boldsymbol{c}) \in W_{k,\ell}$, then $\text{succ}_{k,\ell}(\boldsymbol{c}) = \text{succ}(\boldsymbol{c})$; otherwise, Proposition 1 below states that, $\text{succ}_{k,\ell}(\boldsymbol{c}) = \text{succ}_{p,p}(\boldsymbol{c})$ with $p = ||\boldsymbol{c}||$. Before proving this proposition we need a technical lemma.

Lemma 2. *Let $\boldsymbol{c}, \boldsymbol{c}' \in W_{k,\ell}$ with $\boldsymbol{c}' = \text{succ}_{k,\ell}(\boldsymbol{c})$ and let u be the leftmost position where $c'_i \neq c_i$. Then we have either $c'_u = c_u + 1$ or $c'_u = c_u - 1$.*

Proof. Let $I \subset \{1, 2, \cdots n\}$ be the set of indices i with $c'_i \neq c_i$. So, u is the minimal element of I and suppose that $c'_u = c_u + p$ for some $p > 1$; the case where $c'_u = c_u - p$ is similar.
• If $c'_i > c_i$ for all $i \in I$, then the tuple $\boldsymbol{c}''$ defined by

$$c''_i = \begin{cases} c_i & \text{if} \quad i \neq u, \\ c_i + 1 & \text{if} \quad i = u, \end{cases}$$

belongs to $W_{k,\ell}$ and is such that $\boldsymbol{c} \prec \boldsymbol{c}'' \prec \boldsymbol{c}'$ (actually, in this case $\boldsymbol{c}'' = \text{succ}(\boldsymbol{c})$). This is in contradiction with $\boldsymbol{c}' = \text{succ}_{k,\ell}(\boldsymbol{c})$.
• If there exists $v \in I$, $v > u$, such that $c'_v = c_v - r$ for some $r \geq 1$, then the tuple $\boldsymbol{c}''$ defined by

$$c''_i = \begin{cases} c_i & \text{if} \quad i \neq u, v, \\ c_i + (p-1) & \text{if} \quad i = u, \\ c_i - (r-1) & \text{if} \quad i = v, \end{cases}$$

belongs to $W_{k,\ell}$ and is such that $\boldsymbol{c} \prec \boldsymbol{c}'' \prec \boldsymbol{c}'$, which yields again a contradiction. □

Proposition 1. *Let $\boldsymbol{c}, \boldsymbol{c}' \in W_{k,\ell}$ with $\boldsymbol{c}' = \text{succ}_{k,\ell}(c)$. Then one of the two statements below holds.*

(i) $\boldsymbol{c}' = \text{succ}(\boldsymbol{c})$, and so $\boldsymbol{c}$ and $\boldsymbol{c}'$ differ in precisely one position and with difference 1 in this position,

(ii) $||\boldsymbol{c}|| = ||\boldsymbol{c}'||$, *and* $\boldsymbol{c}$ *and* $\boldsymbol{c}'$ *differ in two positions and by* $+1$ *and* -1 *in these positions.*

Proof. Let u be the leftmost position where $\boldsymbol{c}$ differs from $\boldsymbol{c}'$. By Lemma 2, $c'_u = c_u + \alpha$, with $\alpha \in \{-1, 1\}$.
If $\beta = \sum_{i=1}^{u} c_i$ is odd, then

$$\boldsymbol{c} = c_1 c_2 \dots c_u \text{first}(W^{\boldsymbol{b}'}_{k-\beta,\ell-\beta})$$

and

$$\boldsymbol{c}' = c_1 c_2 \dots (c_u + \alpha)\text{first}(W^{\boldsymbol{b}'}_{k+\alpha-\beta,\ell+\alpha-\beta}).$$

with $\boldsymbol{b}' = b_{u+1} b_{u+2} \dots b_n$. In this case, by Lemma 1 (i) we have either

- $\text{first}(W^{\boldsymbol{b}'}_{k-\beta,\ell-\beta}) = \text{first}(W^{\boldsymbol{b}'}_{k+\alpha-\beta,\ell+\alpha-\beta}) = 00\dots0$, and so $\boldsymbol{c}' = \text{succ}(\boldsymbol{c})$ and (i) follows, or
- $\text{first}(W^{\boldsymbol{b}'}_{k-\beta,\ell-\beta})$ differs from $\text{first}(W^{\boldsymbol{b}'}_{k+\alpha-\beta,\ell+\alpha-\beta})$ in a single position and by $-\alpha$ in this position and (ii) follows.

If $\beta = \sum_{i=1}^{u} c_i$ is even, then

$$\boldsymbol{c} = c_1 c_2 \dots c_u \text{last}(W^{\boldsymbol{b}'}_{k-\beta,\ell-\beta})$$

and

$$\boldsymbol{c}' = c_1 c_2 \dots (c_u + \alpha)\text{last}(W^{\boldsymbol{b}'}_{k+\alpha-\beta,\ell+\alpha-\beta}).$$

Now by applying Lemma 1 (ii) the result holds. □

A consequence of the previous proposition and of its proof is the next corollary.

Corollary 1. *Let* $\boldsymbol{c} \in W_{k,\ell}$ *and* $\boldsymbol{c}' = \text{succ}_{k,\ell}(\boldsymbol{c})$.

(i) If $\text{succ}(\boldsymbol{c}) \in W_{k,\ell}$, *then* $\boldsymbol{c}' = \text{succ}(\boldsymbol{c})$,

(ii) If $\text{succ}(\boldsymbol{c}) \notin W_{k,\ell}$, *then* $\boldsymbol{c}' = \text{succ}_{p,p}(\boldsymbol{c})$ *with* $p = ||\boldsymbol{c}||$. *In this case* $p = k$ *or* $p = \ell$, *and if* u *and* v, $u < v$, *are the two positions where* $\boldsymbol{c}$ *and* $\boldsymbol{c}'$ *differ, then* $c_i = c'_i \in \{0, b_i\}$ *for all* $i > u$, $i \neq v$.

Combining Proposition 1 and Corollary 1, we have:

Theorem 1. *The list* $\mathcal{W}_{k,\ell}$ *of tuples in the set* $W_{k,\ell}$ *listed in* $\prec$ *order is a Gray code where two consecutive tuples differ in at most two positions and by* 1 *in these positions. In particular,* $\mathcal{W}_{k,k}$ *is a Gray code for the* $\boldsymbol{b}$*-bounded compositions of the integer* k.

An alternative Gray code for $W_{k,\ell}$ is given by the next corollary.

Corollary 2. *For* $k \leq \ell \leq ||\boldsymbol{b}||$, *the list*

$$\mathcal{W}_{k,k},\ \overline{\mathcal{W}_{k+1,k+1}},\ \mathcal{W}_{k+2,k+2},\ \cdots,\ \mathcal{W}'_{\ell,\ell}$$

is a Gray code for the set $W_{k,\ell}$, *where* $\mathcal{W}'_{\ell,\ell}$ *is* $\mathcal{W}_{\ell,\ell}$ *or* $\overline{\mathcal{W}_{\ell,\ell}}$ *according as* $\ell - k + 1$ *is odd or even.*

Proof. By Theorem 1, for each i, $k \leq i \leq \ell$, $\mathcal{W}_{i,i}$ is a Gray code for $W_{i,i}$. The last tuple of $\mathcal{W}_{i,i}$ is $\text{last}(W_{i,i})$ and the first tuple of $\overline{\mathcal{W}_{i+1,i+1}}$ is $\text{last}(W_{i+1,i+1})$, and by Lemma 1 (ii) they differ in precisely one position and with difference 1 in this position. Similarly, by Lemma 1 (i) the last tuple of $\overline{\mathcal{W}_{i,i}}$ and the first tuple of $\mathcal{W}_{i+1,i+1}$ differ in the same way. □

For example, for $\boldsymbol{b} = 4\,4 \in \mathbb{N}^2$, we have the following Gray code lists for the set $W_{3,4}$:

- $\mathcal{W}_{3,4} = 03, 04, 13, 12, 21, 22, 31, 30, 40,$ and
- $\mathcal{W}_{3,3}, \overline{\mathcal{W}_{4,4}} = 03, 12, 21, 30, 40, 31, 22, 13, 04$.

In the first list there are 3 transitions of size 2; in the second one there are 7 transitions of size 2. However, the list $04, 03, 13, 12, 22, 21, 31, 30, 40$ for the same set is more restrictive since there is no transition of size 2, and it can be considered 'more optimal'. The existence of 'minimal-change lists' in the general case remains an open problem, see the Acknowledgment at the end of this paper.

As mentioned earlier, the set $W_{k,\ell}$ generalizes the notions of product set, unrestricted and bounded compositions of an integer. The next remark says that this remains true in the ordered case.

Remark 2. As particular cases we have:

- $\mathcal{W}_{0,||\boldsymbol{b}||}$ is the Reflected Gray Code, $\mathcal{G}_n(\boldsymbol{b})$, for the product set $[\boldsymbol{b}]$ defined by (1) (cf. Er in [1]), and which is generated looplessly by Williamson in [16, p. 112],
- if $\boldsymbol{b} = [k]^n$, then $\mathcal{W}_{k,k}$ is Knuth's [16, 7] Gray code for unrestricted n-compositions of k,
- $\mathcal{W}_{k,k}$ becomes Walsh's Gray code for $\boldsymbol{b}$-bounded n-compositions of k, defined and generated looplessly in [13].

For two lists $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \subset \mathcal{B}$ means that $\mathcal{A}$ is a (possibly scattered) sublist of $\mathcal{B}$; in this case the corresponding subsets satisfy $A \subset B$. With this notation we have

Remark 3. $W^{\boldsymbol{b}}_{u,v} \subset W^{\boldsymbol{c}}_{k,\ell}$ if $[u, v]$ is a sub-interval of $[k, \ell]$ and $\boldsymbol{b}$ is componentwise smaller than or equal to $\boldsymbol{c}$. In particular, $\mathcal{W}^{\boldsymbol{b}}_{k,k} \subset \mathcal{W}^{\boldsymbol{b}}_{k,\ell} \subset \mathcal{W}^{\boldsymbol{b}}_{0,||\boldsymbol{b}||} = \mathcal{G}_n(\boldsymbol{b})$.

4 Restricted permutations

There is a natural correspondence between the product set $[0] \times [1] \times \cdots \times [n-1]$ and the set S_n of length-n permutations. Let $\mathcal{G}_n([0] \times [1] \times \cdots \times [n-1])$ be the previous defined Gray code for the product set $[0] \times [1] \times \cdots \times [n-1]$ and $\mathcal{S}_n$ its *list-image* through this correspondence. Lemma 4 says that $\mathcal{S}_n$ is a Gray code for S_n, which is actually the well known Johnson-Trotter-Steinhaus Gray code for permutations [6, 11, 10]. Moreover, Proposition 2 and 3 say that the restriction of $\mathcal{S}_n$ to some particular classes of permutations yields still a Gray code.

In a permutation $\tau \in S_n$ a couple (i, j) is an inversion if $i < j$ but $\tau(i) > \tau(j)$. The array $\boldsymbol{t} = t_1 t_2 \ldots t_n \in [0] \times [1] \times \cdots \times [n-1]$ is the *inversion table* (see [9, p. 20]) of a permutation $\tau \in S_n$ if, for any i, $1 \le i \le n$,

$$t_i = \text{the number of elements in } \tau \text{ smaller than } i \text{ and at its right.}$$

Clearly, $||\boldsymbol{t}||$ is the number of inversions in the permutation τ, denoted $\operatorname{inv} \tau$; it is also the number of adjacent transpositions needed to sort the permutation τ.

$\mathcal{G}_n(\boldsymbol{b})$	$\mathcal{W}_{2,4}$		
	$\mathcal{W}_{2,3}$		$\mathcal{W}_{4,4}$
	$\mathcal{W}_{2,2}$	$\mathcal{W}_{3,3}$	
000			
001			
002	✓		
003		✓	
013			✓
012		✓	
011	✓		
010			
110	✓		
111		✓	
112			✓
113			
103			✓
102		✓	
101	✓		
100			
200	✓		
201		✓	
202			✓
203			
213			
212			
211			✓
210		✓	

Table 1: The sets $[\boldsymbol{b}]$, $W_{2,2}$, $W_{3,3}$, $W_{2,3}$, $W_{4,4}$, $W_{2,4}$ listed in $\prec$ order for $n = 3$ and $\boldsymbol{b} = 2\,1\,3 \in \mathbb{N}^3$.

For every $n \in \mathbb{N}$, the function

$$\phi : [0] \times [1] \times \cdots \times [n-1] \to S_n$$

defined by $\tau = \phi(\boldsymbol{t})$ where $\boldsymbol{t}$ is the inversion table of τ, is a bijection from $[0] \times [1] \times \cdots \times [n-1]$ into S_n.

If two permutations differ by an adjacent transposition, then their inversion tables differ in precisely one position and with difference 1 in this position. Conversely, it is easy to see that if two inversion tables differ in precisely one position and with difference 1 in this position, then their corresponding permutations differ by the transposition of two elements, which are not necessarily adjacent. For example, $\boldsymbol{s} = 011032$ differs from $\boldsymbol{t} = 001032$ in a single position and $253614 = \phi(\boldsymbol{s})$ differs from $153624 = \phi(\boldsymbol{t})$ by a non adjacent transposition. However, under additional constraints, the adjacency property is preserved.

Lemma 3. *Let $\boldsymbol{s}, \boldsymbol{t} \in [0] \times [1] \times [2] \times [n-1]$. If there is an $i \in \{1, 2, \ldots, n\}$ with:*

- *$t_i = s_i + 1$ and $s_j = t_j$ for all $j \neq i$,*
- *$s_j = t_j \in \{0, j-1\}$ for all $j > i$,*

then $\sigma = \phi(\boldsymbol{s})$ and $\tau = \phi(\boldsymbol{t})$ differ by an adjacent transposition.

Proof. If $i = n$, then n is not on the leftmost position in σ, and τ is obtained from σ by transposing n and the element at its left.
If $i \neq n$, then n is the leftmost or rightmost element of σ according as s_n is $n-1$ or 0, and generally, any j, $j > i$, is the leftmost or rightmost element of the permutation obtained from σ by deleting all elements larger than j. So, σ has the form

$$\sigma = \underbrace{\sigma_1 \sigma_2 \ldots \sigma_u}_{>i} \underbrace{\sigma_{u+1} \sigma_{u+2} \ldots \sigma_{u+i}}_{\leq i} \underbrace{\sigma_{u+i+1} \sigma_{u+i+2} \ldots \sigma_n}_{>i}$$

with $\sigma_{u+1}\sigma_{u+2}\ldots\sigma_{u+i}$ a permutation in S_i with the inversion table $s_1 s_2 \ldots s_i$. Since $s_i \neq i-1$, i is not the leftmost element of $\sigma_{u+1}\sigma_{u+2}\ldots\sigma_{u+i}$, and (as in the case $i = n$) τ is obtained from σ by transposing i and the element at its left. $\square$

We say that two permutations differ by an *adjacent transposition* if one can be obtained from the other by transposing two adjacent elements.

By the previous lemma and Lemma 1 (iii) we have:

Lemma 4. *If $\boldsymbol{s}$ and $\boldsymbol{t}$ are two successive tuples in $\mathcal{G}_n([0] \times [1] \times \cdots \times [n-1])$ then the permutations $\phi(\boldsymbol{s})$ and $\phi(\boldsymbol{t})$ in S_n differ by an adjacent transposition, and so $\mathcal{S}_n$ is a Gray code for S_n.*

Actually $\mathcal{S}_n$ is classical Johnson-Trotter-Steinhaus Gray code for the set of length n permutations [6, 11, 10].

Proposition 2. *The restriction of $\mathcal{S}_n$ to the set of permutations with a number of inversions lying between two integers is a Gray code where two consecutive permutations differ by one or two adjacent transpositions.*

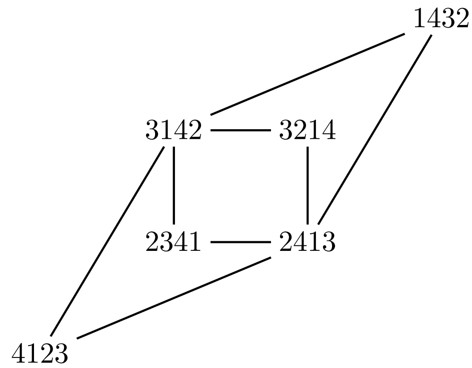

Figure 1: The graph with vertex set the permutations in S_4 with 3 inversions and two permutations are connected if they differ in three positions.

Proof. Let σ and τ be two consecutive permutations in the restriction of $\mathcal{S}_n$ to the set of permutations with a number of inversions between k and ℓ. Let $\boldsymbol{s}$ and $\boldsymbol{t}$ be the corresponding tuples in $\boldsymbol{b} = [0] \times [1] \times \cdots \times [n-1]$ with $\phi(\boldsymbol{s}) = \sigma$ and $\phi(\boldsymbol{t}) = \tau$. By the definition of $\mathcal{S}_n$, $\boldsymbol{s}$ and $\boldsymbol{t}$ are consecutive in $\mathcal{W}^{\boldsymbol{b}}_{k,\ell}$. So, by Proposition 1 either: $\boldsymbol{t} = \mathrm{succ}^{\boldsymbol{b}}(\boldsymbol{s})$ and in this case, by Lemma 4, σ differs from τ by an adjacent transposition; or $\boldsymbol{s}$ and $\boldsymbol{t}$ differ in two positions, say u and v, $u < v$, and by $+1$ and -1 in these positions. In this last case according to Corollary 1 (ii) $s_i = t_i \in \{0, i-1\}$ for all $i > u$ and $i \neq v$. We will show that σ and τ differ by two adjacent transpositions.

Let σ' and π' be the permutations in S_{v-1} with the transposition table $s_1 s_2 \ldots s_{v-1}$ and $t_1 t_2 \ldots t_{v-1}$. By Lemma 3, σ' and π' differ by an adjacent transposition. Now, the permutations σ'' and π'' in S_v with the transposition table $s_1 s_2 \ldots s_{v-1} s_v$ and $t_1 t_2 \ldots t_{v-1} t_v$ differ by two adjacent transpositions. Indeed, σ'' is obtained from σ' by inserting v in the s_vth position from right to left, and τ'' is the permutation obtained from τ' by inserting v in the t_vth position from right to left and, the rightmost position being position zero. Now, since $s_i = t_i \in \{0, i-1\}$ for $i > v$, it results that σ and τ differ as σ'' and τ'', that is by two adjacent transpositions.

□

The previous proposition says that two consecutive permutations in the restriction of $\mathcal{S}_n$ to the set of permutations with a number of inversions lying between two integers differ in at most four positions. Remark that in general there is no more restrictive Gray code for this set. Indeed, an example is given by the graph in Figure 1 which is not Hamiltonian (recall that a Gray code corresponds to a Hamiltonian path in the induced graph).

A *cycle* in a permutation $\pi \in S_n$ is a sequence $(a_0 a_1 \ldots a_{j-1})$ such that $\pi(a_i) = a_{(i+1) \bmod j}$ for all i, $0 \leq i \leq j-1$. Any permutation is the union of disjoint cycles. For example, the permutation $\pi = 4\,2\,5\,1\,7\,6\,3 \in S_7$ is the union of four cycles, namely $(4\,1)$, (2), $(5\,7\,3)$ and (6).

A permutation is called *even* (resp. *odd*) if it has an even (resp. odd) number of inversions. The set of even permutations forms a subgroup of S_n denoted by A_n and it is called the alternating group. Its cardinality is $\frac{n!}{2}$.

Proposition 3. *The restriction of $\mathcal{S}_n$ to the following sets yields Gray codes where two consecutive permutations differ by two adjacent transpositions:*

1. *The set of even permutations;*

2. *The set of odd permutations;*

3. *The set of permutations with an even number of cycles;*

4. *The set of permutations with an odd number of cycles.*

Proof. For the point 1 and 2 the proof is based on the following remark: if $\boldsymbol{s}$ is the successor of $\boldsymbol{r}$, and $\boldsymbol{t}$ that of $\boldsymbol{s}$, in the list $\mathcal{G}_n([0] \times [1] \times \cdots \times [n-1])$, then $||\boldsymbol{r}||$ and $||\boldsymbol{t}||$ have the same parity. The permutations $\phi(\boldsymbol{r})$ and $\phi(\boldsymbol{t})$ differ by two adjacent transpositions, and have the same parity.

For the point 3 and 4 the proof is similar to that of the point 1 and 2 and is based on the following remark: a transposition (not necessarily adjacent) in a permutation glues two cycles in a single one, or splits one cycle in two ones. $\square$

Acknowledgment

The authors would like to thank one of the anonymous referees for helpful suggestions and for providing the open problem following Corollary 2.

References

[1] M.C. Er, On generating the N-ary reflected Gray code, *IEEE Transaction on computers*, **33**(8), 739–741, 1984.

[2] G. Blin, M. Crochemore, S. Hamel and S. Vialette, Finding the median of three permutations under the Kendall-τ distance, *Permutation Patterns 2009* Firenze, 13-17 July 2009, 31–36.

[3] S. Effler, F. Ruskey, A CAT algorithm for generating permutations with a fixed number of inversions, *Information Processing Letters*, **86**, 107-112, 2003.

[4] F. Gray, Pulse code communication, U.S. Patent 2632058 (1953).

[5] J. Joichi, D. E. White and S. G. Williamson, Combinatorial Gray codes, *Siam J. on Computing*, **9**, 130–141, 1980.

[6] S.M. Johnson, Generation of permutations by adjacent transpositions, *Math. Comp.*, **17**, 282–285, 1963.

[7] P. Klingsberg, A Gray Code for compositions, *Journal of Algorithms*, 3 (**1**), 41–44, 1981.

[8] D.E. Knuth, *The Art of Computer Programming*, vol 4, Addison Wesley, 2005.

[9] R. Stanley, *Enumerative Combinatorics*, Vol. 1, Cambridge University Press, Cambridge, England, 1997.

[10] H. Steinhaus, *One hundred problems in elementary mathematics*, Dover Publications 1979 (In Polish 1958).

[11] H.F. Trotter, Perm (Algorithm 115), *Comm. ACM*, **5**(8), 434–435, 1962.

[12] V. Vajnovszki and T. Walsh, A loop-free two-close Gray code algorithm for listing k-ary Dyck words, *Journal of Discrete Algorithms*, **4**(4), 633-648, 2006.

[13] T. Walsh, Loop-free sequencing of bounded integer compositions, *Journal of Combinatorial Mathematics and Combinatorial Computing*, **33**, 323-345, 2000.

[14] T. Walsh, Generating Gray codes in $O(1)$ worst-case time per word, *4th Discrete Mathematics and Theoretical Computer Science Conference*, Dijon-France, 7-12 July 2003 (LNCS 2731, 71–88).

[15] H. S. Wilf, *Combinatorial algorithms: an update*, SIAM, Philadelphia, 1989.

[16] S. G. Williamson, *Combinatorics for computer science*, Computer Science Press, Rockville, Maryland 1985.

[17] A.J. van Zanten, The ranking problem of a Gray code for compositions, *Ars Combinatoria*, **41**, 257–268, 1995.