Catalan words avoiding pairs of length three patterns

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Catalans words are particular growth-restricted words over the set of non-negative integers: the word $\boldsymbol{w} = w_1 w_2 \ldots w_n$ is called Catalan word if $w_1 = 0$ and $0 \leq w_i \leq w_{i-1} + 1$ for $i = 2, 3, \ldots, n$. We denote by $\mathcal{C}_n$ the set of length n Catalan words, and $|\mathcal{C}_n|$ is the nth Catalan number, see for instance [4, exercise 6.19.u, p. 222].

A pattern $\pi = p_1 p_2 \ldots p_k$ is said to be contained in the word $\boldsymbol{w}$ if there is a sub-sequence of $\boldsymbol{w}$, $w_{i_1} w_{i_2} \ldots w_{i_k}$, order-isomorphic with $p_1 p_2 \ldots p_k$. If $\boldsymbol{w}$ does not contain π, we say that $\boldsymbol{w}$ avoids π. If π is a pattern (a set of patterns), $\mathcal{C}_n(\pi)$ denotes the words in $\mathcal{C}_n$ avoiding π (each pattern in π), and $c_n(\pi) = |\mathcal{C}_n(\pi)|$.

Sequel of the work initiated in [1] where, among other things, Catalan words avoiding a length three pattern are enumerated, in this article we almost complete the enumeration of Catalan words avoiding *a pair* of length three patterns (the remaining difficult case is left as open problem) and obtain the Wilf classification of these pairs of patterns. Some of the resulting enumerating sequences are not yet recorded in [3] .

Our methods include structural characterization, recurrence relations, constructive bijections and (bivariate) generating functions. For some pairs of patterns we give the descent distribution and popularity on the set of Catalan words avoiding these patterns.

At the end of this introductory part, we recall results from [1] summarized in Table 1.

Pattern π	Sequence $c_n(\pi)$	Generating function	OEIS
012, 001, 010	2^{n-1}	$\frac{1-x}{1-2x}$	A011782
021	$(n-1)\cdot 2^{n-2}+1$	$\frac{1-4x+5x^2-x^3}{(1-x)(1-2x)^2}$	A005183
102, 201	$\frac{3^{n-1}+1}{2}$	$\frac{1-3x+x^2}{(1-x)(1-3x)}$	A007051
120, 101	F_{2n-1}	$\frac{1-2x}{1-3x+x^2}$	A001519
011	$\frac{n(n-1)}{2}+1$	$\frac{1-2x+2x^2}{(1-x)^3}$	A000124
000	–	$\frac{1-2x^2}{1-x-3x^2+x^3}$	–
100	$\left\lceil\frac{(1+\sqrt{3})^{n+1}}{12}\right\rceil$	$\frac{1-2x-x^2+x^3}{1-3x+2x^3}$	A057960
110	$\frac{1}{2}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n+1}{2k+1}2^k-\frac{n-1}{2}$	$\frac{1-3x+2x^2+x^3}{(1-x)^2(1-2x-x^2)}$	–
210	–	$\frac{1-5x+7x^2-x^3-x^4}{(1-2x)(1-4x+3x^2+x^3)}$	–

Table 1: The cardinality of $\mathcal{C}_n(\pi)$ for each pattern π of length three.

Avoiding a length two and a length three pattern

Proposition For $n \geq 3$ we have:

- $c_n(00,\sigma) = \begin{cases} 0 & \text{if } \sigma = 012, \\ 1 & \text{otherwise.} \end{cases}$
- $c_n(01,\sigma) = \begin{cases} 0 & \text{if } \sigma = 000, \\ 1 & \text{otherwise.} \end{cases}$
- $c_n(10,\sigma) = \begin{cases} F_{n+1} & \text{if } \sigma = 000, \\ n & \text{if } \sigma \in \{001, 011, 012\}, \\ 2^{n-1} & \text{otherwise.} \end{cases}$

Avoiding two length three patterns

Our enumerating results for Catalan words avoiding a pairs of patterns of length three are encompassed in Table 2. The enumeration is given either by a closed expression or by a (bivariate) generating function according to the length (and number of descents). As a byproduct, we give the descent distribution and the descent popularity on the set of Catalan words avoiding some of these pairs of patterns.

References

[1] J.L. Baril, S. Kirgizov and V. Vajnovszki, Descent distribution on Catalan words avoiding a pattern of length at most three.

[2] S. Kitaev. *Patterns in permutations and words*. Springer, 2011.

[3] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, `http://oeis.org/`.

[4] R.P. Stanley. *Enumerative Combinatorics*, volume 2. Cambridge University Press, 1999.

$\pi \setminus \sigma$	000	001	010	100	011	110	101	012	021	102	201	120	210
000	Table 1	F_{n+1}	F_{n+1}	$c_n(000)$	1,2,3,3,3	NEW	2^{n-1}	1,2,3,3,0,0	NEW	NEW	NEW	NEW	NEW
001	-	2^{n-1}	n	$F_n - 1$	n	$\frac{n(n-1)}{2}+1$	2^{n-1}	n	$\frac{n(n-1)}{2}+1$	2^{n-1}	2^{n-1}	$\frac{n(n-1)}{2}+1$	$\binom{n+1}{3}+n+1$
010	-	-	2^{n-1}	2^{n-1}	n	2^{n-1}	2^{n-1}	n	2^{n-1}	2^{n-1}	2^{n-1}	2^{n-1}	2^{n-1}
100	-	-	-	A057960	$2(n+1)$	NEW	Pell number	$\frac{n(n-1)}{2}+1$	$2^n - n$	?	F_{2n-1}	A034943	A267905
011	-	-	-	-	$\frac{n(n-1)}{2}+1$	$\frac{n(n-1)}{2}+1$	$\frac{n(n-1)}{2}+1$	n	$\frac{n(n-1)}{2}+1$	$\frac{n(n-1)}{2}+1$	$\frac{n(n-1)}{2}+1$	$2(n-1)$	$\frac{n(n-1)}{2}+1$
110	-	-	-	-	-	Table 1	$2^n - n$	$\frac{n(n-1)}{2}+1$	NEW	A116702	NEW	A034943	NEW
101	-	-	-	-	-	-	F_{2n-1}	$\frac{n(n-1)}{2}+1$	$2^n - n$	F_{2n-1}	F_{2n-1}	$2^n - n$	$(n-1)2^{n-2}+1$
012	-	-	-	-	-	-	-	2^{n-1}	2^{n-1}	2^{n-1}	2^{n-1}	2^{n-1}	2^{n-1}
021	-	-	-	-	-	-	-	-	$(n-1)2^{n-2}+1$	A116702	$(n-1)2^{n-2}+1$	A045623	$(n-1)2^{n-2}+1$
102	-	-	-	-	-	-	-	-	-	$\frac{3^{n-1}+1}{2}$	NEW	$(n-1)2^{n-2}+1$	NEW
201	-	-	-	-	-	-	-	-	-	-	$\frac{3^{n-1}+1}{2}$	F_{2n-1}	NEW
120	-	-	-	-	-	-	-	-	-	-	-	F_{2n-1}	F_{2n-1}
210	-	-	-	-	-	-	-	-	-	-	-	-	Table 1

Table 2: Number of Catalan words avoiding two pattern of length three.