Exhaustive generation of some classes of pattern avoiding permutations using succession functions

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Outline

- Introduction:
  - pattern avoiding permutations
  - exhaustive generating algorithms
  - succession functions

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  - pattern avoiding permutations = object
  - exhaustive generating algorithms = goal
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  - pattern avoiding permutations = object
  - exhaustive generating algorithms = goal
  - succession functions = tool
- Results
  - χ
  - generic generating algorithm
  - list of good classes of pattern avoiding permutations

Pattern avoiding permutations

- $\mathfrak{S}_n$ is the set of length n permutations

Pattern avoiding permutations

- $\mathfrak{S}_n$ is the set of length n permutations
- $\alpha \in \mathfrak{S}_n$ *contains* $\tau \in \mathfrak{S}_k$ if there is a subsequence

$$1 \leq i_1 < i_2 < \cdots < i_k \leq n$$

such that

$$(\alpha_{i_1}, \ldots, \alpha_{i_k})$$

is order-isomorphic to τ (= *pattern*)

4 7 2 5 6 3 1

3 2 1

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$$
\begin{array}{ccccccccccc}
 & & & & & & & & 3 & 2 & 1 \\
4 & 7 & 2 & 5 & 6 & 3 & 1 & \longrightarrow & 7 & 5 & 3
\end{array}
$$

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$$\begin{array}{cccccccccc} & & & & & & & & 3 & 1 & 2 \\ 4 & 7 & 2 & 5 & 6 & 3 & 1 & \longrightarrow & 4 & 2 & 3 \end{array}$$

- $\mathfrak{S}_n(\tau)$ is the set of permutations in $\mathfrak{S}_n$ avoiding τ

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- for a set A of permutations, $\mathfrak{S}_n(A)$ is the set of permutations in $\mathfrak{S}_n$ avoiding each permutation in A


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pattern avoiding permutations
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Environ 355 000 résultats (0,26 secondes)
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Plus d'outils
Conseil : Recherchez des résultats uniquement en français. Vous pouvez indiquer votre langue de recherche sur
page Préférences.
Articles universitaires correspondant aux termes pattern avoiding permutations
Four classes of pattern-avoiding permutations under ... - Bousquet-Mélou - Cité 53 fois
Permutations with restricted patterns and Dyck paths - Krattenthaler - Cité 136 fois
Pattern Avoiding Permutations III - [Traduire cette page]
6 Feb 2005 ... The third international conference on Permutation Patterns will take place at
the University of Florida, in Gainesville, Florida, March 7-11 ...
math.haifa.ac.il/toufik/conf.../pp05.html - En cache - Pages similaires
[PDF] Pattern-Avoiding Permutations Steven Finch April 27, 2006 Let σ ... - [Traduire cette page
Format de fichier: PDF/Adobe Acrobat - Afficher
Pattern-Avoiding Permutations. 4 which is the unique positive zero of 1 + 2x + x2 + x3 ... to
pattern-avoiding permutations,. Formal Power Series and Al- ...
algo.inria.fr/csolve/av.pdf - Pages similaires
Permutation Pattern -- from Wolfram MathWorld - [Traduire cette page]
30 Aug 2010 ... The following table gives the numbers of pattern-avoiding permutations of
{1,...,n} for various sets of patterns. ...
mathworld.wolfram.com › ... › Permutations - En cache - Pages similaires

- Catalan

  $\mathfrak{S}(321)$, $\mathfrak{S}(312)$

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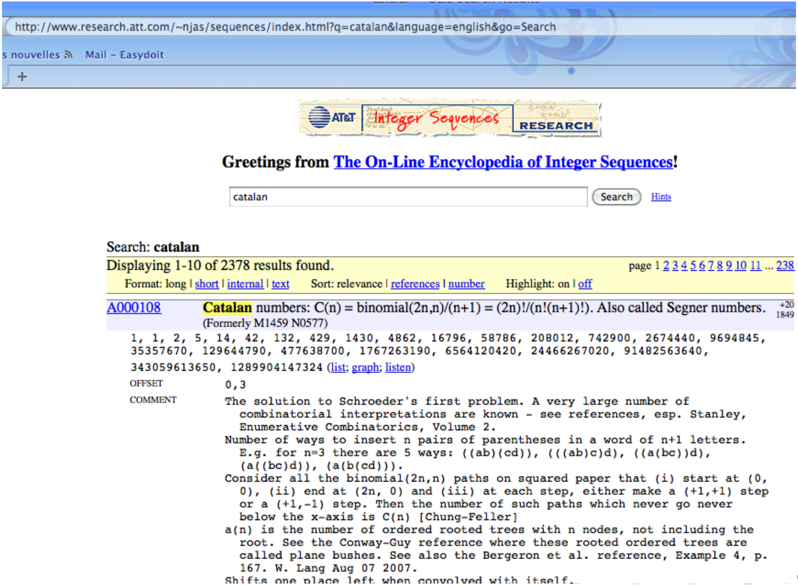

http://www.research.att.com/~njas/sequences/index.html?q=catalan&language=english&go=Search

s nouvelles Mail – Easydoit

AT&T Integer Sequences RESEARCH

Greetings from The On-Line Encyclopedia of Integer Sequences!

catalan Search Hints

Search: **catalan**

Displaying 1-10 of 2378 results found. page 1 2 3 4 5 6 7 8 9 10 11 ... 238

Format: long | short | internal | text Sort: relevance | references | number Highlight: on | off

A000108 **Catalan** numbers: C(n) = binomial(2n,n)/(n+1) = (2n)!/(n!(n+1)!). Also called Segner numbers. +20 1849
(Formerly M1459 N0577)

```
1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845,
35357670, 129644790, 477638700, 1767263190, 6564120420, 24466267020, 91482563640,
343059613650, 1289904147324 (list; graph; listen)
```

OFFSET 0,3

COMMENT
```
The solution to Schroeder's first problem. A very large number of
  combinatorial interpretations are known - see references, esp. Stanley,
  Enumerative Combinatorics, Volume 2.
Number of ways to insert n pairs of parentheses in a word of n+1 letters.
  E.g. for n=3 there are 5 ways: ((ab)(cd)), (((ab)c)d), ((a(bc))d),
  (a((bc)d)), (a(b(cd))).
Consider all the binomial(2n,n) paths on squared paper that (i) start at (0,
  0), (ii) end at (2n, 0) and (iii) at each step, either make a (+1,+1) step
  or a (+1,-1) step. Then the number of such paths which never go never
  below the x-axis is C(n) [Chung-Feller]
a(n) is the number of ordered rooted trees with n nodes, not including the
  root. See the Conway-Guy reference where these rooted ordered trees are
  called plane bushes. See also the Bergeron et al. reference, Example 4, p.
  167. W. Lang Aug 07 2007.
Shifts one place left when convolved with itself.
```

- Fibonacci numbers

  $\mathfrak{S}(321, 312, 231)$

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  $\mathfrak{S}(321, 3412, 4123)$, $\mathfrak{S}(312, 4321, 3421)$

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- ...

An exhaustive generating algorithm for a class of combinatorial objects is an algorithm that produces exhaustively (with no repetition nor omissions) the objects of the class

If a generating algorithm produces combinatorial objects so that only a constant amount of computation is done between successive objects, in an amortized sense, then one says that it runs in constant amortized time (CAT).

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A recursive generating algorithm satisfying the following properties is a CAT algorithm (Ruskey):

The amount of computation in each call is proportional to the number of recursive calls produced by it, and each call

1. is a terminal call and produces a combinatorial object, or
2. produces at least two recursive calls, or

Succession rules

- The *sites* of $\pi \in \mathfrak{S}_n$ are the positions between two consecutive entries, before the first and after the last entry; they are numbered, from right to left, from 1 to $n+1$

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$$(k) \rightsquigarrow (\chi_T(1, k))(\chi_T(2, k)) \ldots (\chi_T(k, k))$$

$$\text{or } (k) \rightsquigarrow \cup_{i=1}^{k}(\chi_T(i, k)), \text{ for } k \geq 1,$$

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• succession function $\rightarrow$ succession rule
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$T = \{312\}$

1

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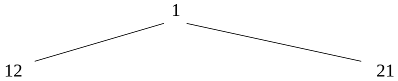
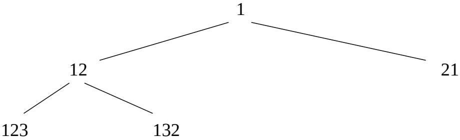
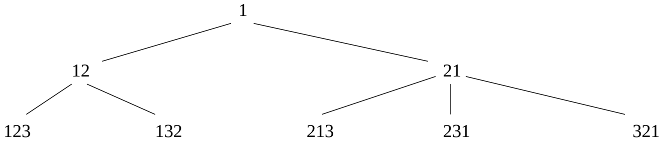


$T = \{312\}$



$T = \{312\}$



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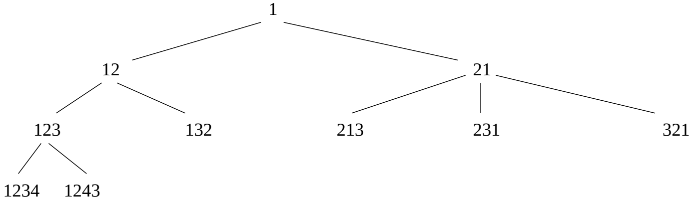

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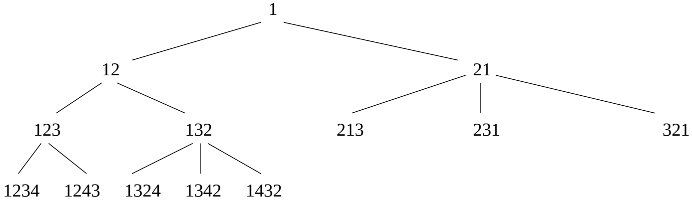

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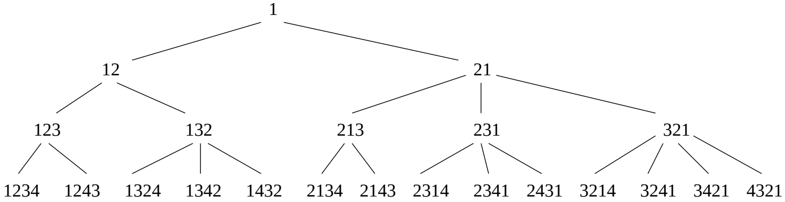
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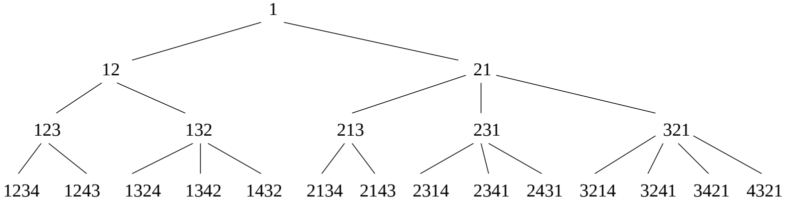
$T = \{312\}, \chi_T(i,k) = i+1$



$T = \{321\}$

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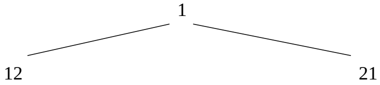

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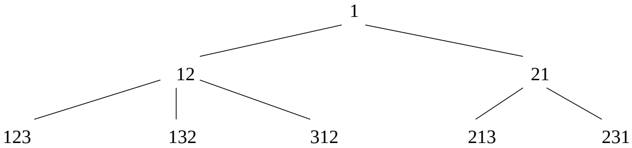

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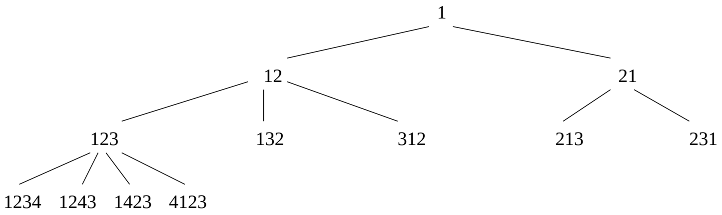

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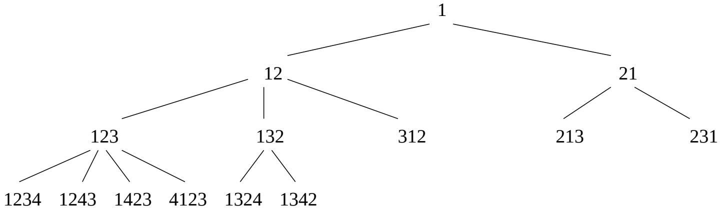

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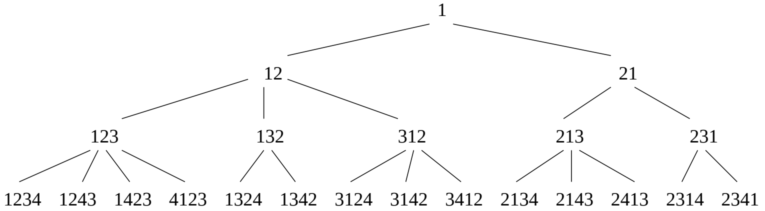
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$$T = \{321\}\ \chi_T(i,k) = \begin{cases} k+1 & \text{if } i = 1 \\ i & \text{otherwise} \end{cases}$$

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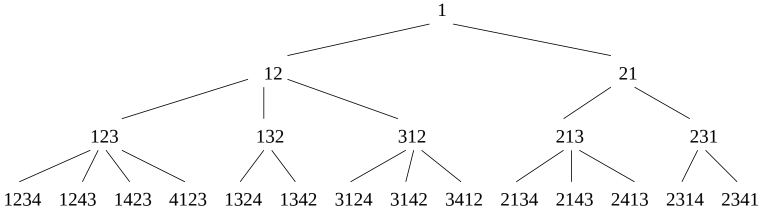

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```
procedure Gen_Avoid(size, k)
local i
if size = n then Print(π)
else size := size + 1
     π := [π, size]
     Gen_Avoid(size, χ(1, k))
     for i := 2 to k do
          π := π · (size − i + 2, size − i + 1)
          Gen_Avoid(size, χ(i, k))
     end do
     for i := k dowto 2 do
          π := π · (size − i + 2, size − i + 1)
     end do
end if
end procedure.
```

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Combinatorial Gray codes for classes of pattern avoiding permutations

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Abstract

The past decade has seen a flurry of research into pattern avoiding permutations but little of it is concerned with their exhaustive generation. Many applications call for exhaustive generation of permutations subject to various constraints or imposing a particular generating order. In this paper we present generating algorithms and combinatorial Gray codes for several families of pattern

- 2^{n-1}
  - $T = \{321, 312\}$, $\chi_T(i,k) = 2$
  - $T = \{321, 231\}$, $\chi_T(i,k) = \begin{cases} k+1 & \text{if } i = 1 \\ 1 & \text{otherwise} \end{cases}$
- Pell numbers
  - $T = \{321, 3412, 4123\}$, $\chi_T(i,k) = \begin{cases} 3 & \text{if } i = 1 \\ 2 & \text{otherwise} \end{cases}$
  - $T = \{312, 4321, 3421\}$, $\chi_T(i,k) = \begin{cases} 3 & \text{if } i = 2 \\ 2 & \text{otherwise} \end{cases}$

- Even index Fibonacci numbers
  - $T = \{321, 3412\}$, $\chi_T(i,k) = \begin{cases} k+1 & \text{if } i = 1 \\ 2 & \text{otherwise} \end{cases}$
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  - $T = \{312, 4321\}$, $\chi_T(i,k) = \begin{cases} 3 & \text{if } k = 3 \text{ and } i = 3 \\ i+1 & \text{otherwise} \end{cases}$

- Catalan numbers
  - $T = \{312\}$, $\chi_T(i,k) = i+1$
  - $T = \{321\}$, $\chi_T(i,k) = \begin{cases} k+1 & i=1 \\ i & \text{otherwise} \end{cases}$

- Schröder numbers
  - $T = \{1234, 2134\}$, $\chi_T(i,k) = \begin{cases} k+1 & i=1 \text{ or } i=2 \\ i & \text{otherwise} \end{cases}$
  - $T = \{1324, 2314\}$, $\chi_T(i,k) = \begin{cases} k+1 & i=1 \text{ or } i=k \\ i+1 & \text{otherwise} \end{cases}$
  - $T = \{4123, 4213\}$, $\chi_T(i,k) = \begin{cases} k+1 & i=k-1 \text{ or } i=k \\ i+2 & \text{otherwise} \end{cases}$

- Grand Dyck numbers
  - $T = \{1234, 1324, 2134, 2314\}$,
    $$\chi_T(i,k) = \begin{cases} k+1 & i=1 \\ 3 & i=2 \\ i & \text{otherwise} \end{cases}$$
  - $T = \{1324, 2314, 3124, 3214\}$,
    $$\chi_T(i,k) = \begin{cases} 3 & i=1 \\ i+1 & \text{otherwise} \end{cases}$$

- Motzkin numbers
    - $T=\{321,3\bar{1}42\}$, $\chi_T(i,k)=\begin{cases} k+1 & i=1 \\ i-1 & \text{otherwise}\end{cases}$
- numbers of left factors of Motzkin words
    - $T=\{321,4\bar{1}523\}$, $\chi_T(i,k)=\begin{cases} k+1 & i=1 \\ 2 & i=2 \\ i-1 & \text{otherwise}\end{cases}$
- Fibonacci numbers
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HVALA !