A new Euler-Mahonian constructive bijection

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Abstract

Using generating functions, MacMahon proved in 1916 the remarkable fact that the major index has the same distribution as the inversion number for multiset permutations, and in 1968 Foata gave a constructive bijection proving MacMahon's result. Since then, many refinements have been derived, consisting of adding new constraints or new statistics.

Here we give a new simple constructive bijection between the set of permutations with a given number of inversions and those with a given major index. We introduce a new statistic, mix, related to the Lehmer code, and using our new bijection we show that the bistatistic $(\mathsf{mix}, \mathsf{INV})$ is Euler-Mahonian. Finally we introduce the McMahon code for permutations which is the major-index counterpart of the Lehmer code and show that the two codes are related by a simple relation.

1 Preliminaries

We say that i is a *descent* of $\pi \in \mathfrak{S}_n$ if $\pi_i > \pi_{i+1}$ and the *descent set* of π is the set of its descents. The pair (i,j) is an *inversion* of $\pi \in \mathfrak{S}_n$ if $i<j$ but $\pi_i > \pi_j$. A *statistic* on $\mathfrak{S}_n$ is an association of an element of $\mathbb{N}$ to each permutation in $\mathfrak{S}_n$ and a bistatistic is a pair of statistics. For a permutation $\pi \in \mathfrak{S}_n$ the *descent number* des, *major index* MAJ, *inversion number* INV are statistics defined by (see, for example, [6, Section 10.6])

$$\mathsf{des}\,\pi = \operatorname{card}\{i \mid 1 \le i \le n-1, \pi_i > \pi_{i+1}\},$$

$$\mathsf{MAJ}\,\pi = \sum_{\substack{1\le i<n \\ \pi_i>\pi_{i+1}}} i,$$

$$\mathsf{INV}\,\pi = \operatorname{card}\{(i,j) \mid 1 \le i < j \le n, \pi_i > \pi_j\}.$$

and thus, $\mathsf{INV}\,\pi$ equals the number of inversions of π.

This note is a revised form of the preliminary conference version [9] and the main results are Theorems 7, 9 and 13, in Sections 2, 3 and 4, respectively. Theorem 7 establishes a bijection between the set of permutations with a given number of inversions and those with a given major index. Theorem 9 relates the descent statistic des with a new statistic mix introduced here, from which it follows (Corollary 10) that the bistatistic $(\mathsf{mix}, \mathsf{INV})$ is Euler-Mahonian. Finally, we introduce the McMahon code and Theorem 13 relates the Lehmer code with the McMahon code.

2 Lehmer's code and the bijections ϕ and ψ

An integer sequence $t_1t_2\dots t_n$ is said to be *subexcedent* if $0 \le t_i \le i-1$ for $1 \le i \le n$, and the set of length-n subexcedent sequences is denoted by S_n; so $S_n = \{0\} \times \{0,1\} \times \dots \times \{0,1,\dots,n-1\}$. The *Lehmer code* [5] is a bijection $code : \mathfrak{S}_n \to S_n$ which maps each permutation $\pi = \pi_1\pi_2\dots\pi_n$ to a subexcedent sequence $t_1t_2\dots t_n$ where, for all i, $1 \le i \le n$, t_i is the number of inversions (j,i) in π (or equivalently, the number of entries in π larger than π_i and on its left). In this case, $\mathsf{INV}\,\pi = \sum_{i=1}^{n} t_i$.

Let permutations act on indices, i.e., for $\sigma = \sigma_1\,\sigma_2\,\dots\,\sigma_n$ and $\tau = \tau_1\,\tau_2\,\dots\,\tau_n$ two permutations in $\mathfrak{S}_n$, $\sigma \cdot \tau = \sigma_{\tau_1}\,\sigma_{\tau_2}\,\dots\,\sigma_{\tau_n}$. For a fixed integer n, let k and u be two integers, $0 \le k < u \le n$, and define $\rho_{u,k} \in \mathfrak{S}_n$ as the permutation obtained from the identity in $\mathfrak{S}_n$ after a left circular shift of the segment of length $k+1$ ending at position u. In two line notation we have

$$\rho_{u,k} = \begin{pmatrix} 1 & 2 & \cdots & u-k-1 & u-k & u-k+1 & \cdots & u-1 & u & u+1 & \cdots & n \\ 1 & 2 & \cdots & u-k-1 & u-k+1 & u-k+2 & \cdots & u & u-k & u+1 & \cdots & n \end{pmatrix}.$$

For example, in $\mathfrak{S}_5$ we have: $\rho_{3,1} = 1\,\underline{3\,2}\,4\,5$, $\rho_{3,2} = \underline{2\,3\,1}\,4\,5$ and $\rho_{5,3} = 1\,\underline{3\,4\,5\,2}$ (the rotated elements are underlined). Since the permutation $\rho_{u,k}$ has its first $u-1$ entries in increasing order so does the permutation $\rho_{n,t_n} \cdot \rho_{n-1,t_{n-1}} \cdot \ldots \cdot \rho_{j,t_j} = \prod_{i=n}^{j} \rho_{i,t_i}$, for all j, $1 \le j \le n$, and $0 \le t_i \le i-1$. Thus we have

Remark 1. *The function $t_1t_2\dots t_n \mapsto \prod_{i=n}^{1} \rho_{i,t_i}$ is the inverse of $code : \mathfrak{S}_n \to S_n$ and so a bijection from S_n onto $\mathfrak{S}_n$.*

Therefore, every permutation $\pi \in \mathfrak{S}_n$ can be recovered from its Lehmer code $t = t_1t_2\dots t_n \in S_n$ by (see Figure 1 (a) for an example)

$$\begin{aligned} \pi &= \rho_{n,t_n} \cdot \rho_{n-1,t_{n-1}} \cdot \ldots \cdot \rho_{i,t_i} \cdot \ldots \cdot \rho_{2,t_2} \cdot \rho_{1,t_1} \\ &= \prod_{i=n}^{1} \rho_{i,t_i}. \end{aligned}$$

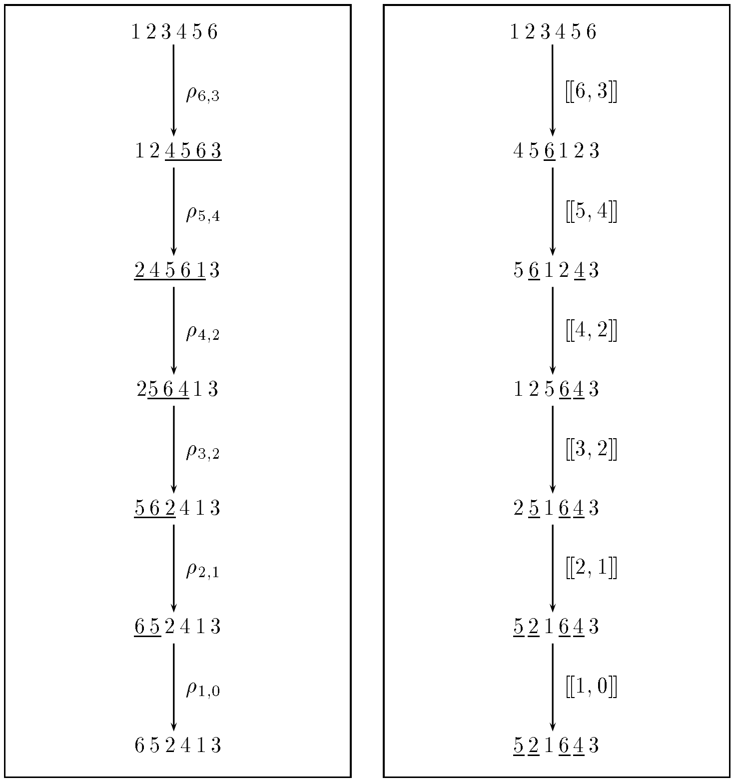
Clearly $\mathsf{INV} \prod_{i=n}^{1} \rho_{i,t_i} = \sum_{i=1}^{n} t_i$.

Let n, k and u be three integers as above ($0 \le k < u \le n$) and define $[\![u,k]\!] \in \mathfrak{S}_n$ as the permutation obtained after k right circular shifts of the length-u prefix of the identity in $\mathfrak{S}_n$. In two line notation we have

$$[\![u,k]\!] = \begin{pmatrix} 1 & 2 & \cdots & k & k+1 & \cdots & u & u+1 & \cdots & n \\ u-k+1 & u-k+2 & \cdots & u & 1 & \cdots & u-k & u+1 & \cdots & n \end{pmatrix}.$$

For example, in $\mathfrak{S}_5$ we have: $[\![3,1]\!] = \underline{3\,1\,2}\,4\,5$, $[\![3,2]\!] = \underline{2\,3\,1}\,4\,5$ and $[\![5,3]\!] = \underline{3\,4\,5\,1\,2}$ (the rotated elements are underlined). Obviously, $[\![u,p]\!] \cdot [\![u,r]\!] = [\![u,p+r]\!]$, with addition taken modulo u.

Let $\psi : S_n \to \mathfrak{S}_n$ be the function defined by



(a) (b)

Figure 1: (a) The construction of the permutation $\rho_{6,3} \cdot \rho_{5,4} \cdot \rho_{4,2} \cdot \rho_{3,2} \cdot \rho_{2,1} \cdot \rho_{1,0} = 6\,5\,2\,4\,1\,3 \in \mathfrak{S}_6$ having the Lehmer code $t = 0\,1\,2\,2\,4\,3$; rotated segments are underlined. (b) The construction of the permutation $\psi(012243) = [[6,3]] \cdot [[5,4]] \cdot [[4,2]] \cdot [[3,2]] \cdot [[2,1]] \cdot [[1,0]] = 5\,2\,1\,6\,4\,3 \in \mathfrak{S}_6$ from the identity by successive prefix rotations; in each permutation descents are underlined. The permutation obtained in (b) is the image through ϕ of the one obtained in (a).

$$\begin{aligned}\psi(t_1 t_2 \dots t_n) &= [[n, t_n]] \cdot [[n-1, t_{n-1}]] \cdot \dots \cdot [[i, t_i]] \cdot \dots \cdot [[2, t_2]] \cdot [[1, t_1]] \\ &= \prod_{i=n}^{1} [[i, t_i]].\end{aligned}$$

(See Figure 1 (b) for an example.)

The next lemma says that every permutation in $\mathfrak{S}_n$ can be uniquely written as $\prod_{i=n}^{1}[[i, t_i]]$ for some t_i's. Thus, $\{\rho_{i,k}\}_{0 \le k < i \le n}$ and $\{[[i,k]]\}_{0 \le k < i \le n}$ are both generating sets for $\mathfrak{S}_n$; and, as we will show later, $[[i,k]]$ can be viewed as a 'MAJ counterpart' of $\rho_{i,k}$.

Lemma 1. *The function ψ defined above is a bijection.*

Proof. Firstly, ψ is an injective function. Indeed, let $s = s_1 s_2 \dots s_n$ and $t = t_1 t_2 \dots t_n$ be two sequences in S_n with $s \neq t$ and let $\sigma = \psi(s)$ and $\tau = \psi(t)$. If j is the rightmost position with $s_j \neq t_j$, then we have:

- $\prod_{i=n}^{j+1}[[i, s_i]] = \prod_{i=n}^{j+1}[[i, t_i]]$, and

- $\prod_{i=n}^{j} [\![i, s_i]\!] \neq \prod_{i=n}^{j} [\![i, t_i]\!]$.

For $i < j$, $[\![i, s_i]\!]$ and $[\![i, t_i]\!]$ act only on the first i entries of permutations in $\mathfrak{S}_n$ and so $\sigma_j \neq \tau_j$. Finally, cardinality considerations show that ψ is a bijection. $\square$

The map $\phi : \mathfrak{S}_n \to \mathfrak{S}_n$ defined by $\phi(\pi) = \psi(t)$ with t being the Lehmer code of π is a bijection and $\phi = \psi \circ code$, see Figure 2. Now we introduce the notion of *k-separate permutations* and give a technical lemma.

$$\begin{tikzcd} \mathfrak{S}_n \arrow[r, "code"] \arrow[dr, "\phi"'] & S_n \arrow[d, "\psi"] \\ & \mathfrak{S}_n \end{tikzcd}$$

Figure 2: $\phi = \psi \circ code$.

Definition 2. *We say that $\pi \in \mathfrak{S}_n$ is* k-separate, *$1 \leq k \leq n$, if there exists an ℓ such that π can be written as the concatenation of three 'segments' (the first of them possibly empty)*

$$\pi = \underbrace{\pi_1\pi_2\ldots\pi_\ell}\,\underbrace{\pi_{\ell+1}\pi_{\ell+2}\ldots\pi_{k-1}\pi_k}\,\underbrace{\pi_{k+1}\ldots\pi_n} \tag{1}$$

with

- *$\pi_i < \pi_j$ for all i and j, $1 \leq i < j \leq \ell$ or $\ell + 1 \leq i < j \leq k$, and*
- *$\pi_i > \pi_j$ for all i and j, $1 \leq i \leq \ell < j \leq k$.*

Figure 4 shows the matrix representation of a separate permutation. Note that if π is k-separate, then π has at most one descent to the left of k, and (with the notations above) this descent is ℓ, if there is any. Also, every k-separate permutation is also j-separate for $1 \leq j < k$ and every permutation is 1-separate.

Lemma 3. *Let $\pi \in \mathfrak{S}_n$ be k-separate, $1 < k \leq n$. For an i and a v with $0 < v < i < k$ let σ denote the permutation $\pi \cdot [\![i, v]\!]$.*

a) If π has no descents to the left of k, then v is the unique descent in σ to the left of k.

Otherwise, let ℓ be the (unique) descent in π to the left of k. In this case:

b) if $v \leq i - \ell$, then $\ell + v$ is the unique descent in σ to the left of k;

c) if $v > i - \ell$, then σ has two descents to the left of k, namely i and $v - i + \ell$.

Proof. In the case *a)* the shape of π is

$$\pi = \underbrace{\pi_1\pi_2\ldots\pi_{k-1}\pi_k}\,\underbrace{\pi_{k+1}\ldots\pi_n}$$

with $\pi_i < \pi_j$ for $1 \leq i < j \leq k$. In this case v is a descent of σ (with $\sigma_v = \pi_i$) and no other descent is produced and so the case *a)* holds.

Now suppose that ℓ is the (unique) descent in π to the left of k. Thus π is the concatenation of three segments, as in relation (1) of Definition 2.

When $v \leq i - \ell$, then σ has a single descent to the left of k, namely $\ell + v$ (with $\sigma_{\ell+v} = \pi_\ell$); and the case *b)* holds.

Finally, when $v > i - \ell$, then σ has two descents to the left of k: i (with $\sigma_i = \pi_{i-v}$) and $v - i + \ell$ (with $\sigma_{v-i+\ell} = \pi_\ell$), and the case *c)* holds. $\square$

The following two corollaries are consequences of the previous lemma.

```
r := 0;
for i := n downto 1 do
    r := r + t_i;
    if i ≤ r
    then b_i := 1; r := r − i;
    else b_i := 0;
    endif
enddo
```

Figure 3: Algorithm computing the characteristic vector $b = b_1 b_2 \dots b_{n-1}$ of the descent set of $\pi = \prod_{i=n}^{1} [\![i, t_i]\!]$.

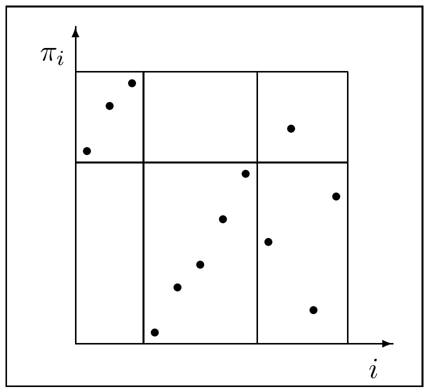


Figure 4: The permutation 9 11 12 1 3 4 6 8 5 10 2 7 $\in \mathfrak{S}_{12}$ is 8-separate (and so j-separate for $1 \leq j < 8$).

Corollary 4. *For a given $t = t_1 t_2 \dots t_n \in S_n$, the algorithm in Figure 3 computes in linear time the* characteristic vector *of the descent set of $\psi(t) \in \mathfrak{S}_n$; that is, the binary vector b with $b_i = 1$ if and only if i is a descent of $\psi(t)$.*

Proof. After each iteration on i, b_i is set to 1 if and only if i is a descent of $\psi(t)$, and $r \neq 0$ is the leftmost descent in the permutation $\prod_{j=n}^{i} [\![j, t_j]\!]$. $\square$

In Table 1 there are a few examples where the characteristic vectors of the descent sets are given in the second column.

In particular, Lemma 3 gives the following corollaries.

Corollary 5. *If π is k-separate, $k > 1$, then for all i and v, $0 \leq v < i < k$, $\pi \cdot [\![i, v]\!]$ is i-separate. In addition $\mathsf{MAJ}\,(\pi \cdot [\![i, v]\!]) = \mathsf{MAJ}\,\pi + v$.*

Corollary 6. *For every $t = t_1 t_2 \dots t_n \in S_n$, we have $\mathsf{MAJ} \prod_{i=n}^{1} [\![i, t_i]\!] = \sum_{i=1}^{n} t_i$.*

Proof. By iteratively applying the previous corollary to $\prod_{j=n}^{i} [\![j, t_j]\!]$. $\square$

Theorem 7. *For every $\pi \in \mathfrak{S}_n$, we have $\mathsf{MAJ}\,\phi(\pi) = \mathsf{INV}\,\pi$.*

Proof. Let $t = t_1 t_2 \dots t_n$ be the Lehmer code of π. By definition $\phi(\pi) = \prod_{i=n}^{1} [\![i, t_i]\!]$ and, applying the above corollary the statement holds.

$\square$

A consequence of the previous theorem is the following well-known result [7, 3]:

Corollary 8. $\sum_{\sigma\in\mathfrak{S}_n} q^{\mathsf{MAJ}\,\sigma} = \sum_{\sigma\in\mathfrak{S}_n} q^{\mathsf{INV}\,\sigma}$, *that is, the statistics* MAJ *and* INV *are equidistributed on* $\mathfrak{S}_n$*, and so both are Mahonian.*

3 The mix statistic

For a sequence $t = t_1t_2\dots t_n \in S_n$, let $b = b_1b_2\dots b_{n-1}$ be a binary sequence with $\sum_{j=1}^{n-1} j\cdot b_j = \sum_{j=1}^{n} t_j$. Obviously, b is not uniquely determined by t, but if we impose the condition that $b_1, b_2, \dots, b_{n-1}$ satisfy

$$\sum_{j=i}^{n} t_j - i < \sum_{j=i}^{n-1} j\cdot b_j \le \sum_{j=i}^{n} t_j$$

for all $i \ge 1$, then b becomes unique and we call this binary sequence the *multi-radix (binary) array of* t. Note that the same multi-radix array can correspond to several subexcedent arrays, see Table 1.

Now we define the statistic mix on S_n and on $\mathfrak{S}_n$. For $t \in S_n$, $\mathsf{mix}\, t$ is the number of 1-bits in its multi-radix array; and by extension, for $\pi \in \mathfrak{S}_n$ we define $\mathsf{mix}\,\pi = \mathsf{mix}\, t$, where t is the Lehmer code of π. Formally, $\mathsf{mix}\,\pi = \mathsf{mix}\, code(\pi)$ and we extend the des and mix statistics to set-valued functions.

Let $\mathsf{D} : \mathfrak{S}_n \to 2^{\{1,2,\dots,n-1\}}$ be the set-valued function which maps a permutation to its descent set ($2^{\{1,2,\dots,n-1\}}$ denotes the set of all subsets of $\{1, 2, \dots, n-1\}$). $\mathsf{D}(\pi)$ is the descent set of π and we have

$$\mathsf{des}\,\pi = \mathrm{card}\,\mathsf{D}(\pi),$$

$$\mathsf{MAJ}\,\pi = \sum_{i\in\mathsf{D}(\pi)} i.$$

Similarly, define $\mathsf{M} : \mathfrak{S}_n \to 2^{\{1,2,\dots,n-1\}}$ as the set of positions of 1-bits in the multi-radix array of the Lehmer code of π. By definition we have

$$\mathsf{mix}\,\pi = \mathrm{card}\,\mathsf{M}(\pi),$$

$$\mathsf{INV}\,\pi = \sum_{i\in\mathsf{M}(\pi)} i.$$

Theorem 9. *For every subset* T *of* $\{1, 2, \cdots, n-1\}$*, we have*

$$\mathrm{card}\{\pi \in \mathfrak{S}_n \,|\, \mathsf{M}(\pi) = T\} = \mathrm{card}\{\tau \in \mathfrak{S}_n \,|\, \mathsf{D}(\tau) = T\}.$$

Proof. It is easy to check that the multi-radix array of the sequence $t = t_1t_2\dots t_n$ in S_n is precisely the characteristic vector b of the descent set of $\prod_{i=n}^{1}[[i, t_i]] \in \mathfrak{S}_n$ computed by the algorithm in Figure 3; and in this algorithm $r = \sum_{j=i}^{n} t_j - \sum_{j=i}^{n-1} j\cdot b_j$ after each iteration on i. It follows that $\mathsf{M}(\pi) = \mathsf{D}(\phi(\pi))$ for all $\pi \in \mathfrak{S}_n$, and the statement of the theorem holds. □

In particular, we have

Corollary 10. *The statistic* mix *is Eulerian, that is, it has the same distribution as* des*.*

t	multi-radix sequence of t	$\psi(t)$
0 0 0 1 4	1 0 0 1	$\underline{5}\,2\,3\,\underline{4}\,1$
0 0 0 2 3	1 0 0 1	$\underline{5}\,1\,3\,\underline{4}\,2$
0 0 0 3 2	1 0 0 1	$\underline{5}\,1\,2\,\underline{4}\,3$
0 0 1 0 4	1 0 0 1	$\underline{4}\,2\,3\,\underline{5}\,1$
0 0 1 1 3	1 0 0 1	$\underline{4}\,1\,3\,\underline{5}\,2$
0 0 1 2 2	1 0 0 1	$\underline{4}\,1\,2\,\underline{5}\,3$
0 0 1 3 1	1 0 0 1	$\underline{3}\,1\,2\,\underline{5}\,4$
0 0 2 0 3	0 1 1 0	$4\,\underline{5}\,\underline{3}\,1\,2$
0 0 2 1 2	0 1 1 0	$4\,\underline{5}\,\underline{2}\,1\,3$
0 0 2 2 1	0 1 1 0	$3\,\underline{5}\,\underline{2}\,1\,4$
0 0 2 3 0	0 1 1 0	$3\,\underline{4}\,\underline{2}\,1\,5$
0 1 0 0 4	1 0 0 1	$\underline{3}\,2\,4\,\underline{5}\,1$
0 1 0 1 3	1 0 0 1	$\underline{3}\,1\,4\,\underline{5}\,2$
0 1 0 2 2	1 0 0 1	$\underline{2}\,1\,4\,\underline{5}\,3$
0 1 0 3 1	1 0 0 1	$\underline{2}\,1\,3\,\underline{5}\,4$
0 1 1 0 3	0 1 1 0	$3\,\underline{5}\,\underline{4}\,1\,2$
0 1 1 1 2	0 1 1 0	$2\,\underline{5}\,\underline{4}\,1\,3$
0 1 1 2 1	0 1 1 0	$2\,\underline{5}\,\underline{3}\,1\,4$
0 1 1 3 0	0 1 1 0	$2\,\underline{4}\,\underline{3}\,1\,5$
0 1 2 0 2	0 1 1 0	$1\,\underline{5}\,\underline{4}\,2\,3$
0 1 2 1 1	0 1 1 0	$1\,\underline{5}\,\underline{3}\,2\,4$
0 1 2 2 0	0 1 1 0	$1\,\underline{4}\,\underline{3}\,2\,5$

Table 1: The 22 sequences $t = t_1t_2t_3t_4t_5$ in S_5 with $t_1 + t_2 + t_3 + t_4 + t_5 = 5$, their corresponding multi-radix sequences and images $\psi(t)$ with $\mathsf{MAJ}\,\psi(t) = 5$. The positions where the multi-radix array equals 1 are precisely the descents of the corresponding permutation.

A bistatistic is *Euler-Mahonian* if it has the same joint distribution as $(\mathsf{des}, \mathsf{MAJ})$. A consequence of Theorem 9 is

Corollary 11. *The bistatistic* $(\mathsf{mix}, \mathsf{INV})$ *is Euler-Mahonian, or equivalently,*

$$\sum_{\sigma\in\mathfrak{S}_n} t^{\mathsf{mix}\,\sigma} q^{\mathsf{INV}\,\sigma} = \sum_{\sigma\in\mathfrak{S}_n} t^{\mathsf{des}\,\sigma} q^{\mathsf{MAJ}\,\sigma}.$$

So, the statistic mix can be seen just as a new 'Eulerian partner for inversions'; such a 'partner' already exists [8] and it is different from the one presented here. Another Euler-Mahonian bistatistic is $(\mathsf{exc}, \mathsf{den})$, with exc being the excedance number and den the Denert statistic [1, p. 66], thus den is a 'Mahonian partner for the excedance number', see [4].

4 From Lehmer code to McMahon code

Let $\pi \in \mathfrak{S}_n$ and $s = s_1s_2\ldots s_n$ be the subexcedent sequence such that $\pi = \prod_{i=n}^{1}[[i, s_i]]$. Since s is related to MAJ statistic (see Corollary 6) introduced by McMahon in [7] we call s the *McMahon* *code* of π. Equivalently, the McMahon code of π is $\psi^{-1}(\pi)$. Theorem 13 below answers the following question: how are the Lehmer code and McMahon code related? Surprisingly, the techniques involved have both a Gray-code and a data-compression flavor.

Definition 12. *Let $\Delta : S_n \to S_n$ be the function defined by: if $t = t_1 t_2 \dots t_n \in S_n$, then $\Delta(t)$ is the sequence $s_1 s_2 \dots s_n \in S_n$ with*

- $s_n = t_n$, *and*
- $s_i = (t_i - t_{i+1}) \mod i$, *for* $1 \le i \le n-1$.

Clearly, for any t in S_n, $\Delta(t)$ is in S_n, and so Δ is well defined. Also Δ is bijective with the inverse, Δ^{-1}, defined by: if $s_1 s_2 \dots s_n \in S_n$, then $\Delta^{-1}(s)$ is the sequence $t = t_1 t_2 \dots t_n \in S_n$ with

- $t_n = s_n$, and
- $t_i = (t_{i+1} + s_i) \mod i$, for $1 \le i \le n-1$.

For example, in S_6, we have $\Delta(0\,1\,2\,0\,2\,3) = 0\,1\,2\,2\,4\,3$. Figure 1 (b) shows the construction of $5\,2\,1\,6\,4\,3 \in \mathfrak{S}_6$ from its McMahon code $0\,1\,2\,2\,4\,3$.

The function Δ is a slight variation of a well-known transformation in Gray code theory. For an integer k written in binary as $b_1 b_2 \dots b_n$, with the most significant bit b_1, the kth binary sequence in binary reflected Gray code order is $g = g_1 g_2 \dots g_n$, where

- $g_1 = b_1$, and
- $g_i = b_{i-1} \oplus b_i$ for $1 < i \le n$

with $\oplus$ being addition (or, equivalently, substraction) modulo 2. For more details see [2] and the references therein. For instance, for $k = 12 = (1100)_2$, the $12th$ binary sequence in Gray code order is 1010; and for $k = 451 = (111000011)_2$, the $451th$ binary sequence in Gray code order is 100100010. The above transformation $b \mapsto g$ maps runs (consecutive occurrences of the same value) into sequences of 0's preceded by a 1; in the binary case it coincides with *Move To Front* data compression pre-processing transformation.

Theorem 13. *If $\pi \in \mathfrak{S}_n$ has its Lehmer code $t = t_1 t_2 \dots t_n$, then its McMahon code is $\Delta(t)$, that is, $\pi = \prod_{i=n}^{1} [[i, s_i]]$ with $s_1 s_2 \dots s_n = \Delta(t)$.*

Proof. Firstly, note that for three integers n, k and u with $0 \le k < u \le n$,

$$\rho_{u,k} = [[u, k]] \cdot [[u-1, u-1-k]].$$

But $[[v, p]] \cdot [[v, r]] = [[v, p+r]]$, with addition taken modulo v, and the statement holds by iterating this relation for the Lehmer code of π. $\square$

As a consequence of this theorem we have the following corollary and the alternative definition: $\psi = (\Delta \circ code)^{-1}$, see Figure 5.

Corollary 14. *For any $\pi \in \mathfrak{S}_n$, the McMahon code $s = s_1 s_2 \dots s_n$ of π satisfies:*

- $s_i = \mathrm{card}\,\{j \mid 1 \le j < i, \pi_j \in [\pi_i, \pi_{i+1}]\}$ *if* $\pi_i < \pi_{i+1}$,
- $s_i = \mathrm{card}\,\{j \mid 1 \le j < i, \pi_j \notin [\pi_{i+1}, \pi_i]\}$ *elsewhere,*

with the convention that $\pi_{n+1} = n+1$.

For example,

- $\Delta(0\,1\,2\,2\,4\,3) = 0\,1\,0\,2\,1\,3$, and so in $\mathfrak{S}_6$ we have $\rho_{6,3} \cdot \rho_{5,4} \cdot \rho_{4,2} \cdot \rho_{3,2} \cdot \rho_{2,1} \cdot \rho_{1,0} = [[6,3]] \cdot [[5,1]] \cdot [[4,2]] \cdot [[3,0]] \cdot [[2,1]] \cdot [[1,0]] = 6\,5\,2\,4\,1\,3$, and
- $\Delta^{-1}(0\,1\,2\,2\,4\,3) = 0\,1\,2\,0\,2\,3$, and so in $\mathfrak{S}_6$ we have $[[6,3]] \cdot [[5,4]] \cdot [[4,2]] \cdot [[3,2]] \cdot [[2,1]] \cdot [[1,0]] = \rho_{6,3} \cdot \rho_{5,2} \cdot \rho_{4,0} \cdot \rho_{3,2} \cdot \rho_{2,1} \cdot \rho_{1,0} = 5\,2\,1\,6\,4\,3$.

$$\begin{tikzcd} S_n \arrow[r, "\Delta"] \arrow[d, "code^{-1}"'] & S_n \arrow[dl, "\psi"] \\ \mathfrak{S}_n & \end{tikzcd}$$

Figure 5: $\psi = code^{-1} \circ \Delta^{-1} = (\Delta \circ code)^{-1}$.

See again Figure 1.

For a subexcedent sequence $t = t_1 t_2 t_3 \dots t_{n-1} t_n \in S_n$ we define its complement t^c as the subexcedent sequence $t_1 (1 - t_2)(2 - t_3) \dots (n - 2 - t_{n-1})(n - 1 - t_n)$; and for a permutation $\pi = \pi_1 \pi_2 \pi_3 \dots \pi_{n-1} \pi_n \in \mathfrak{S}_n$ we define its complement as $\pi^c = (n + 1 - \pi_1)(n + 1 - \pi_2)(n + 1 - \pi_3) \dots (n + 1 - \pi_{n-1})(n + 1 - \pi_n)$. It is easy to see that for any $t \in S_n$, $code^{-1}(t^c) = (code^{-1}(t))^c$ and we will show that the complement operator commutes with Δ, Δ^{-1} and ψ.

Lemma 15. *For any $t \in S_n$, we have*

(i) $\Delta(t^c) = (\Delta(t))^c$,

(ii) $\Delta^{-1}(t^c) = (\Delta^{-1}(t))^c$,

(iii) $\psi(t^c) = (\psi(t))^c$.

Proof. (i) Let $s = s_1 s_2 \dots s_n = \Delta(t^c)$ and $s' = s'_1 s'_2 \dots s'_n = (\Delta(t))^c$. Clearly, $s_n = s'_n = (n-1) - t_n$ and for $i < n$,

$$\begin{aligned} s_i &= (i - 1 - t_i) - (i - t_{i+1}) \mod i \\ &= t_{i+1} - t_i - 1 \mod i, \end{aligned}$$

and

$$\begin{aligned} s'_i &= i - 1 - (t_i - t_{i+1}) \mod i \\ &= t_{i+1} - t_i - 1 \mod i, \end{aligned}$$

and so $s = s'$.

The proof of point (ii) is similar to that of point (i).

(iii) The statement results from the following equalities:

$$\begin{aligned} \psi(t^c) &= (code^{-1} \circ \Delta^{-1})(t^c) \\ &= code^{-1}(\Delta^{-1}(t))^c \\ &= (code^{-1}(\Delta^{-1}(t)))^c \\ &= (\psi(t))^c. \end{aligned}$$

$\square$

Corollary 16.

- $\mathsf{D}(\psi(t^c)) = \{1, 2, \dots, n-1\} \setminus \mathsf{D}(\psi(t))$,
- $\mathsf{M}(code^{-1}(t)) = \mathsf{D}((code^{-1} \circ \Delta^{-1})(t))$.

Finally, we conclude with the following question: can the previous results be naturally generalized to multiset permutations?

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