Permutation Patterns 2012 – Strathclyde

Lehmer code transforms and Mahonian statistics on permutations

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Outline

- Introduction
  Compressing the Lehmer code for permutations
- Definitions
- Previous work
- Main : Alternative proofs on the *Mahonicity* of some pattern based statistics
- Summary
- Final remarks

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$$\Delta : \{0,1,2\}^8 \to \{0,1,2\}^8$$

$$s_1 s_2 \ldots s_8 = \Delta(t_1 t_2 \ldots t_8)$$

$$s_i = \begin{cases} (t_i - t_{i+1}) \mod 3 & \text{if} \quad 1 \leq i < 8 \\ t_8 & \text{if} \quad i = 8 \end{cases}$$

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$$s_i = \begin{cases} (t_{i-1} - t_i) \mod 3 & \text{if } 2 \le i \le 8 \\ t_1 & \text{if } i = 1 \end{cases}$$

Subexcedant sequence transforms

$$S_n = \{0\} \times \{0,1\} \times \cdots \times \{0,1,\ldots,n-1\}$$

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$u_1 u_2 \cdots u_n = \Gamma(t_1 t_2 \dots t_n)$, then

$$\sum_{i=1}^{n} u_i = \mathrm{STAT}\,\pi$$ (E. Babson, E. Steingrimsson 2000)

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Example

$\pi = 58467123 \in \mathfrak{S}_8$

Lehmer code of π

$$t = 00211555 \xrightarrow{\Delta} s = 00101005$$

$$\text{INV}\,\pi = \sum_{i=1}^{8} t_i = 19 \qquad \text{MAJ}\,\pi = \sum_{i=1}^{8} s_i = 7$$

$$t = 00211555 \xrightarrow{\Gamma} u = 00110200$$

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Previous work

- E. Babson, E. Steingrímsson, 2000 introduced the notion of vincular patterns
  - essentially all well-known Mahonian permutation statistics can be written as combinations of such patterns
  - other combinations of vincular patterns are still Mahonian
- D. Foata, D. Zeilberger 2001, D. Foata, A. Randrianarivony 2006
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Definitions

$\sigma \in \mathfrak{S}_k$ is a (classical) pattern of $\pi \in \mathfrak{S}_n$, $k \leq n$, if there is a sequence

$$1 \leq i_1 < i_2 < \cdots < i_k \leq n$$

such that

$$\pi_{i_1} \pi_{i_2} \cdots \pi_{i_k}$$

is order-isomorphic to σ.

Patterns will be written as words over the alphabet $\{a, b, c, \ldots\}$ based on the usual ordering $a < b < c \cdots$.

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Vincular patterns are generalizations of classical patterns:

- The absence of a $-$ means that the corresponding letters must be adjacent

Example $ca-b$ is a pattern of 652413
$\underline{ca}b$

- Patterns may end with $]$ (begin with $[$) these indicate that they are required to end at the last letter (begin at the first letter) in a permutation.

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A statistic on $\mathfrak{S}_n$

$$\mathfrak{S}_n \to \mathbb{N}$$

For a permutation π and a set of patterns $\{\sigma, \tau, \ldots\}$,

$$(\sigma + \tau + \cdots)\,\pi$$

denotes the number of occurrences of these patterns in π

$$(\sigma + \tau + \cdots)$$

becomes a permutation statistic

Example

$$\mathrm{INV}\,\pi = (b-a)\,\pi,$$

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Pointed pattern is the pattern σ together with a privileged element ℓth one

$$\sigma_1\sigma_2\cdots\underline{\sigma_\ell}\cdots\sigma_k$$

$$(\sigma_1\sigma_2\cdots\underline{\sigma_\ell}\cdots\sigma_k)_i\,\pi$$

denotes the number of occurrences of the pattern σ in the permutation π, where the role of σ_ℓ is played by π_i.

Example if $\pi = 2\,4\,5\,1\,3\,6$, then

- $(b-\underline{a}c)_5\,\pi = 2$
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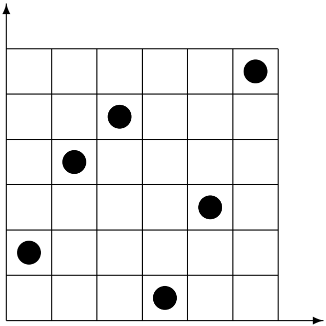

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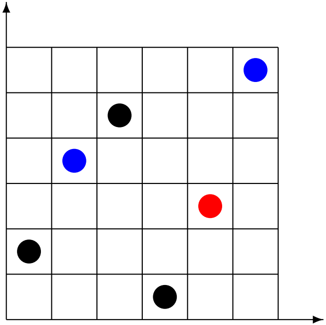

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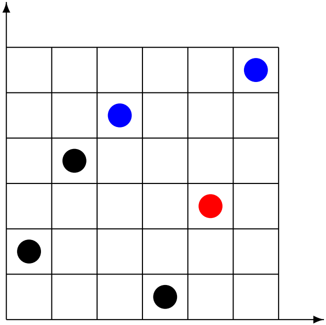

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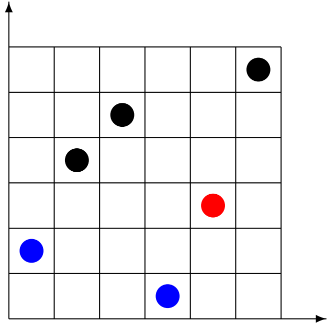

$$\pi = 2\,4\,5\,1\,3\,6$$

$$(b - a\underline{c})_5\,\pi = 1$$

$$(\sigma)\,\pi = \sum_{i=1}^{n} (\underline{\sigma})_i\,\pi$$

for any pointed pattern $\underline{\sigma}$ corresponding to σ

Lehmer's code

Definition

An integer sequence $t_1 t_2 \ldots t_n$ is subexcedent if

$$0 \leq t_i \leq i-1$$

The set of n-length subexcedent sequences is

$$S_n = \{0\} \times \{0,1\} \times \ldots \times \{0,1,\ldots,n-1\}.$$

Definition

A permutation code is a bijection

$$code : \mathfrak{S}_n \to S_n$$

Definition

The Lehmer code is a bijection

$$L : \mathfrak{S}_n \to S_n$$

which maps

$$\pi_1 \pi_2 \ldots \pi_n \mapsto t_1 t_2 \ldots t_n$$

where t_i is the number of inversions (j, i) in π (or equivalently, the number of entries in π larger than π_i and at its left)

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Example

$$\begin{array}{ccccccccc} & \pi & = & 6 & 5 & 2 & 4 & 1 & 3 \\ L(\pi) = & t & = & 0 & 1 & 2 & 2 & 4 & 3 \end{array}$$

$$t_i = (b - \underline{a})_i\,\pi$$

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$$\pi \mapsto t_1 t_2 \cdots t_n$$

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$$t_i = (\underline{\sigma} + \underline{\tau} \cdots)_i \, \pi, \text{ for } 1 \le i \le n,$$

is a permutation code, then we say that the set of patterns $\{\sigma, \tau, \ldots\}$ induces a permutation code

The Lehmer code L is induced both by the set of patterns

- $\{b - a\}$
- $\{c - ba, b - ca, c - ab, ba\}$

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For each permutation code $\pi \mapsto t_1 t_2 \cdots t_n$ we can associate naturally a Mahonian statistic $\textsc{st}$ on $\mathfrak{S}_n$, defined by

$$\textsc{st}\,\pi = \sum_{i=1}^{n} t_i$$

In addition, if the set of patterns $\{\sigma, \tau, \ldots\}$ induces a permutation code, then the statistic

$$\pi \mapsto (\sigma + \tau \cdots)\,\pi$$

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Main results

We call a bijection from S_n onto itself a code transform.

$$\Delta, \Gamma, \Theta, \Lambda, \Upsilon, \Psi : S_n \rightarrow S_n$$

Definition (V. V. 2011)

$$\Delta(t) = s_1 s_2 \cdots s_n$$

$$s_i = \begin{cases} (t_i - t_{i+1}) \mod i & \text{if} \quad 1 \leq i < n \\ t_n & \text{if} \quad i = n. \end{cases}$$

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Proposition

Δ is a code transform

Remark

$$\Delta^{-1}(s) = t_1 t_2 \cdots t_n$$

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$$\begin{array}{rcccccc} L = & 0 & 1 & 1 & 3 & 3 & 3 \\ \Delta(L) = & 0 & 0 & 2 & 0 & 0 & 3 \end{array}$$

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For a permutation $\pi \in \mathfrak{S}_n$ with $L(\pi)$ its Lehmer code, we call $\Delta(L(\pi)) \in S_n$ McMahon code of π.

Proposition (V.V. 2011)

If $s_1 s_2 \cdots s_n$ is the McMahon code of $\pi \in \mathfrak{S}_n$, then

$$\mathrm{MAJ}\,\pi = \sum_{i=1}^{n} s_i$$

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Corollary (V.V. 2011)

For $\pi \in \mathfrak{S}_n$, the McMahon code $s = s_1 s_2 \cdots s_n$ of π is given by:

$$s_i = \begin{cases} ((a - \underline{c}b) + (b - \underline{a}c) + (c - \underline{b}a))_i\, \pi & \text{if} \quad i \neq n \\ (b - a]\, \pi & \text{if} \quad i = n \end{cases}$$

Corollary

For $\pi \in \mathfrak{S}_n$

1. $\mathrm{MAJ}\, \pi = ((a - cb) + (b - ac) + (c - ba) + (b - a])\, \pi$,
2. *If the McMahon code of π is $s_1 s_2 \cdots s_n$, then*

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Remark

If $\sigma, \tau \in \mathfrak{S}_n$ are two permutations with their McMahon codes differing only in the last position, then $\sigma_i \neq \tau_i$ for all i, $1 \leq i \leq n$.

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Proof

If $p_1p_2\cdots p_{n-1}p_n$ is the McMahon code $\pi=\pi_1\pi_2\cdots\pi_n\in\mathfrak{S}_n$ then

$$((a-cb)+(b-ac)+(c-ba)+[b-a))\,\pi=\sum_{i=1}^{n-1}p_i+(\pi_1-1)$$

$$\pi\mapsto p_1p_2\cdots p_{n-1}(\pi_1-1)$$

is an injection and so (by cardinality reasons) a permutation code

$$((a-cb)+(b-ca)+(c-ba)+(ba))\pi = \text{MAJ}\,\pi$$

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Let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$. If $\tau \in \mathfrak{S}_{n-1}$ is the reduction of $\pi_2\cdots\pi_n$, then

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The map $\pi \mapsto s$ is a permutation code.

$$\sum_{i=1}^{n} s_i = ((a-cb)+(b-ca)+(c-ba)+[b-a))\,\pi$$

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$$\Gamma^{-1}(s) = t_1 t_2 \cdots t_n$$

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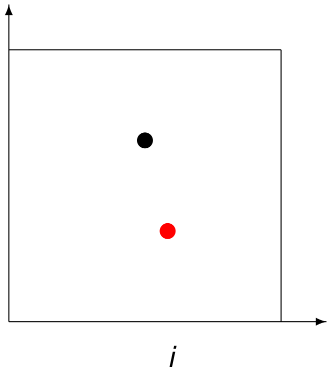
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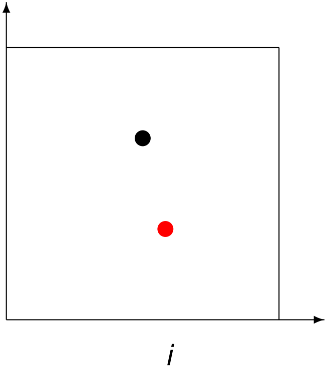

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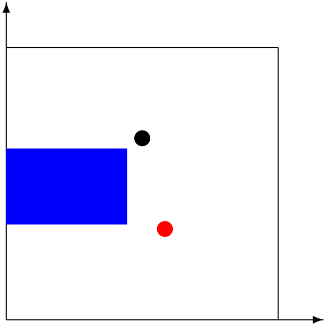

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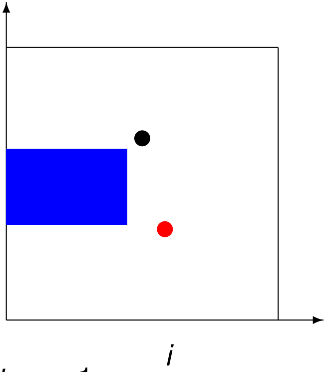

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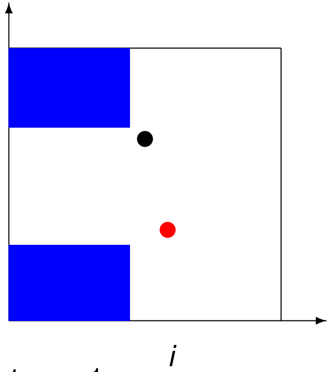

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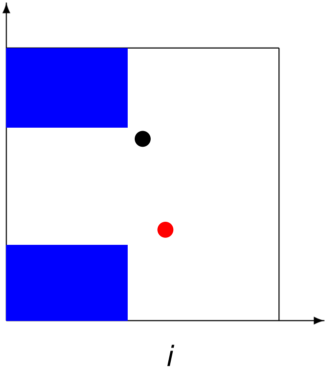

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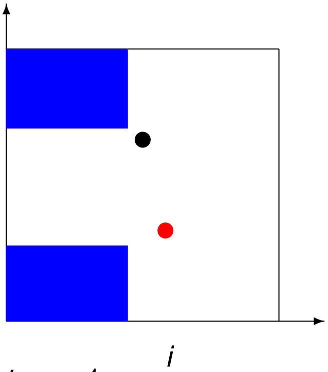

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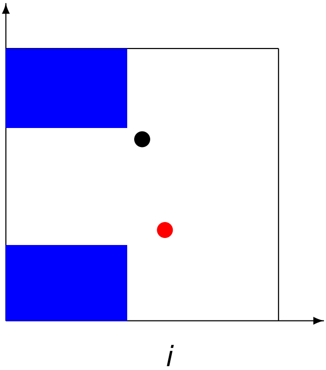

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For any integer n and permutation $\pi \in \mathfrak{S}_n$ we have

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Summary

statistic		transform
INV	$(b-a)$	
	$(b-ca)+(c-ab)+(c-ba)+(ba)$	
MAJ	$(a-cb)+(b-ca)+(c-ba)+(ba)$	
	$(a-cb)+(b-ac)+(c-ba)+(b-a]$	Δ
S_2	$(a-cb)+(b-ac)+(c-ba)+[b-a)$	
S_4	$(a-cb)+(b-ca)+(c-ba)+[b-a)$	
STAT	$(a-cb)+(b-ac)+(c-ba)+(ba)$	Γ
STAT$'$	$(b-ac)+(b-ca)+(c-ba)+(ba)$	Θ
STAT$''$	$(a-cb)+(c-ab)+(c-ba)+(ba)$	Λ
	$(a-cb)+(b-ca)+(b-ca)+(ba)$	Υ
	$(b-ca)+(b-ca)+(c-ab)+(ba)$	Ψ

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$(a - cb)\,\pi = \frac{1}{3} \cdot (\upsilon + 2 \cdot \mathrm{STAT}'' - 2 \cdot \mathrm{INV}\,\pi - \mathrm{des})\,\pi$: $n \log n$

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$(a-cb)\,\pi = \frac{1}{3}\cdot(\upsilon + 2\cdot \text{STAT}'' - 2\cdot \text{INV}\,\pi - \text{des})\,\pi$,
$(b-ac)\,\pi = (\text{STAT}' - \text{MAJ})\,\pi + \frac{1}{3}\cdot(\upsilon + 2\cdot \text{STAT}'' - 2\cdot \text{INV} - \text{des})\,\pi$,
$(b-ca)\,\pi = \frac{1}{3}\cdot(\upsilon + \text{INV} - \text{STAT}'' - \text{des})\,\pi$,
$(c-ab)\,\pi = \frac{1}{3}\cdot(\upsilon + \text{INV} + 2\cdot \text{STAT}'' - \text{des})\,\pi - \text{MAJ}\,\pi$,
$(c-ba)\,\pi = \text{MAJ}\,\pi + \frac{1}{3}\cdot(\text{INV} - 2\cdot\upsilon - \text{STAT}'' - \text{des})\,\pi$, and
$(a-bc)\,\pi = \frac{(n-2)\cdot(n-1)}{2} - x$, with x the sum of the previous five statistics.

Linear dependency

$$\mathrm{STAT}\,\pi = \mathrm{STAT}'\,\pi + \mathrm{STAT}''\,\pi - \mathrm{INV}\,\pi$$

$$S_4\,\pi = S_2\,\pi + \mathrm{STAT}\,\pi - \mathrm{MAJ}\,\pi = (\mathrm{STAT} - \mathrm{des})\,\pi - \pi_1 - 1$$

Constructive bijections

$$\{\pi \in \mathfrak{S}_n \,|\, \mathrm{STAT}\,\pi = k\} \to \{\pi \in \mathfrak{S}_n \,|\, \mathrm{STAT}'\,\pi = k\}$$

defined by

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arxiv.org/abs/1203.3964

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