Lehmer code transforms and Mahonian statistics on permutations

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November 30, 2012

Abstract

In 2000 Babson and Steingrímsson introduced the notion of vincular patterns in permutations. They show that essentially all well-known Mahonian permutation statistics can be written as combinations of such patterns. Also, they proved and conjectured that other combinations of vincular patterns are still Mahonian. These conjectures were proved later: by Foata and Zeilberger in 2001, and by Foata and Randrianarivony in 2006.

In this paper we give an alternative proof of some of these results. Our approach is based on permutation codes which, like Lehmer's code, map bijectively permutations onto subexcedant sequences. More precisely, we give several code transforms (i.e., bijections between subexcedant sequences) which when applied to Lehmer's code yield new permutation codes which count occurrences of some vincular patterns. These code transforms can be seen as a pre-compression step of Lehmer's code because they map some redundancies into runs of 0s. Also, our proofs, unlike the previous ones, provide explicit bijections between permutations having a given value for two different Mahonian pattern-based statistics.

1 Introduction

An alternative way to represent a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$ is by its permutation code $t_1t_2\cdots t_n$, which is a subexcedant sequence. A classical example of permutation code is the Lehmer code, where each t_i is the number of entries in π larger than π_i and on its left. The inv statistic on permutations is related to Lehmer code by $\mathsf{inv}\,\pi = \sum_{i=1}^{n} t_i$.

A code transform is a bijection from the set of subexcedant sequences onto itself. We give several code transforms and show that most pattern-involvement based statistics introduced in [1] are related to transforms of Lehmer code in the same way as INV is related to Lehmer code. These results are summarized in Table 1 at the end of this paper.

1.1 Permutation patterns

A permutation $\sigma \in \mathfrak{S}_k$ is a classical pattern of the permutation $\pi \in \mathfrak{S}_n$, $k \leq n$, if there is a sequence $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ is order-isomorphic to σ. Vincular patterns, introduced in [1] and extensively studied later (see for example [2, 3, 4]), are generalizations of classical patterns where:

- Two adjacent letters may or may not be separated by a dash. The absence of a dash between two adjacent letters means that the corresponding letters in the permutation must be adjacent;

- Patterns may begin and/or end with square brackets. These indicate that they are required to begin at the first letter in a permutation and/or end at the last letter.

In the following patterns will be written as words over the alphabet $\{a, b, c, \ldots\}$ based on the usual ordering $a < b < c \cdots$. For example, in the permutation $4\,3\,2\,6\,1\,5 \in \mathfrak{S}_6$ there are four occurrences of $(ba\text{-}c)$, namely: $4\,3\,6$, $4\,3\,5$, $3\,2\,6$ and $3\,2\,5$; but only $4\,3\,5$ and $3\,2\,5$ are occurrences of $(ba\text{-}c]$.

1.2 Mahonian statistics and pattern involvement

A *statistic* on $\mathfrak{S}_n$ is an association of an integer to each permutation in $\mathfrak{S}_n$. Classical examples of statistics are INV and MAJ defined as

$$\begin{aligned}
\mathsf{INV}\,\pi &= \operatorname{card}\{(i,j) \mid 1 \leq i < j \leq n, \pi_i > \pi_j\},\\
\mathsf{MAJ}\,\pi &= \sum_{\substack{1 \leq i < n \\ \pi_i > \pi_{i+1}}} i.
\end{aligned}$$

A statistic ST on $\mathfrak{S}_n$ is *Mahonian* if it has the same distribution as INV, that is

$$\operatorname{card}\{\pi \in \mathfrak{S}_n \,|\, \mathsf{ST}\,\pi = k\} = \operatorname{card}\{\pi \in \mathfrak{S}_n \,|\, \mathsf{INV}\,\pi = k\},$$

for any $k \geq 0$, and it is well known that MAJ is a Mahonian statistic.

For a permutation π and a set of patterns $\{\sigma, \tau, \ldots\}$, we denote by $(\sigma + \tau + \cdots)\,\pi$ the number of occurrences of these patterns in π, and $(\sigma + \tau + \cdots)$ becomes a permutation statistic. For example

$$\mathsf{INV}\,\pi = (b\text{-}a)\,\pi,$$

and

$$\mathsf{MAJ}\,\pi = ((a\text{-}cb) + (b\text{-}ca) + (c\text{-}ba) + (ba))\,\pi.$$

In order to count the number of occurrences of the pattern σ in π we introduce the notion of *pointed pattern*. A pointed pattern is the pattern σ together with a privileged element, say the ℓth one; and we denote by $\sigma_1\sigma_2\cdots\underline{\sigma_\ell}\cdots\sigma_k$ such a pattern. Often, when the privileged element is understood or does not matter we denote simply by $\underline{\sigma}$ a pointed pattern if the underlying permutation is σ. With these notations, $(\sigma_1\sigma_2\cdots\underline{\sigma_\ell}\cdots\sigma_k)_i\,\pi$ denotes the number of occurrences of the pattern σ in the permutation π, where the role of σ_ℓ is played by π_i. For example, if $\pi = 2\,4\,5\,1\,3\,6$, then $(b\text{-}\underline{a}c)_5\,\pi = 2$, and $(b\text{-}a\underline{c})_5\,\pi = 1$. Clearly,

$$(\sigma)\,\pi = \sum_{i=1}^{n} (\underline{\sigma})_i\,\pi$$

for any pointed pattern $\underline{\sigma}$ corresponding to σ.

1.3 Permutation codes

An integer sequence $t_1t_2\cdots t_n$ is said to be *subexcedant* if $0 \leq t_i \leq i-1$ for $1 \leq i \leq n$, and the set of all length-n subexcedant sequences is denoted by S_n; so $S_n = \{0\} \times \{0,1\} \times \cdots \times \{0,1,\ldots,n-1\}$. Clearly, $\mathfrak{S}_n$ is in bijection with S_n, and any such bijection is called *permutation code*.

The *Lehmer code* [7] is a classical example of permutation code and will be our starting point for the construction of other several permutation codes. This code bijectively maps each permutation onto a subexcedant sequence of same length.

Definition 1. For $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$, the Lehmer code $L(\pi)$ of π is the subexcedant sequence $t_1t_2\cdots t_n$ where, for all i, $1 \leq i \leq n$, t_i is the number of inversions (j,i) in π, that is, number of j with $\pi_j > \pi_i$ but $j < i$.

For example $L(5\,2\,1\,6\,4\,3) = 0\,1\,2\,0\,2\,3$. Some permutation codes can be obtained from pattern occurrences, and this is the case for the Lehmer code $t_1t_2\cdots t_n$ of a permutation π:

$$t_i = (b\text{-}\underline{a})_i\,\pi \qquad (1)$$

and so $\mathsf{INV}\,\pi = (b\text{-}a)\,\pi$. Alternatively,

$$t_i = ((b\text{-}c\underline{a}) + (c\text{-}a\underline{b}) + (c\text{-}b\underline{a}) + (b\underline{a}))_i\,\pi, \qquad (2)$$

and $\mathsf{INV}\,\pi = ((b\text{-}ca) + (c\text{-}ab) + (c\text{-}ba) + (ba))\,\pi$.

Let $\{\underline{\sigma}, \underline{\tau}, \ldots\}$ be a set of pointed patterns. If the function

$$\pi \mapsto t_1 t_2 \cdots t_n$$

where

$$t_i = (\underline{\sigma} + \underline{\tau} \cdots)_i\,\pi, \text{ for } 1 \le i \le n,$$

is a permutation code C, then we say that the set of patterns $\{\sigma, \tau, \ldots\}$ induces the permutation code C. For example, relations (1) and (2) show that the Lehmer code L is induced both by the set of patterns $\{(b\text{-}a)\}$ and $\{(c\text{-}ba), (b\text{-}ca), (c\text{-}ab), (ba)\}$.

For each permutation code $\pi \mapsto t_1 t_2 \cdots t_n$ we can associate naturally a Mahonian statistic ST on $\mathfrak{S}_n$, defined by

$$\mathsf{ST}\,\pi = \sum_{i=1}^{n} t_i.$$

In addition, if the set of patterns $\{\sigma, \tau, \ldots\}$ induces a permutation code, then the statistic

$$\pi \mapsto (\sigma + \tau \cdots)\,\pi$$

is Mahonian. We will see that, like $\{(b\text{-}a)\}$ and $\{(c\text{-}ba), (b\text{-}ca), (c\text{-}ab), (ba)\}$, several other patterns induce permutation codes, and so these patterns are Mahonian, see Table 1.

2 Code transforms

We call a bijection from S_n onto itself a *code transform*. Below we give six functions and we show that they are code transforms, and thus each of them applied to the Lehmer code yields still a permutation code.

Definition 2. The functions $\Delta, \Gamma, \Theta, \Lambda, \Theta, \Upsilon, \Psi : S_n \to S_n$ are defined as follows. If $t = t_1 t_2 \cdots t_n \in S_n$, then

- $\Delta(t) = s_1 s_2 \cdots s_n$ is defined by $s_i = \begin{cases} (t_i - t_{i+1}) \mod i & \text{if} \quad 1 \le i < n \\ t_n & \text{if} \quad i = n. \end{cases}$

- $\Gamma(t) = s_1 s_2 \cdots s_n$ is defined by $s_i = \begin{cases} t_1 & \text{if} \quad i = 1 \\ (t_{i-1} - t_i) \mod i & \text{if} \quad 1 < i \le n. \end{cases}$

- $\Theta(t) = s_1 s_2 \cdots s_n$ is defined by $s_i = \begin{cases} t_1 & \text{if} \quad i = 1 \\ t_{i-1} - t_i & \text{if} \quad t_{i-1} \ge t_i \text{ and } 1 < i \le n \\ t_i & \text{if} \quad t_{i-1} < t_i \text{ and } 1 < i \le n. \end{cases}$

- $\Lambda(t) = s_1 s_2 \cdots s_n$ is defined by $s_i = \begin{cases} t_1 & \text{if} \quad i = 1 \\ t_i & \text{if} \quad t_{i-1} \geq t_i \text{ and } 1 < i \leq n \\ i + t_{i-1} - t_i & \text{if} \quad t_{i-1} < t_i \text{ and } 1 < i \leq n. \end{cases}$

- $\Upsilon(t) = s_1 s_2 \cdots s_n$ is defined by $s_i = \begin{cases} i - t_i - 1 & \text{if} \quad t_i < t_{i+1} \text{ and } 1 \leq i < n \\ t_i - t_{i+1} & \text{if} \quad t_i \geq t_{i+1} \text{ and } 1 \leq i < n \\ t_n & \text{if} \quad i = n. \end{cases}$

- $\Psi(t) = s_1 s_2 \cdots s_n$ is defined by $s_i = \begin{cases} t_{i+1} - t_i - 1 & \text{if} \quad t_i < t_{i+1} \text{ and } 1 \leq i < n \\ t_i & \text{if} \quad t_i \geq t_{i+1} \text{ and } 1 \leq i < n \\ t_n & \text{if} \quad i = n. \end{cases}$

See Example 1 and Figure 1.a for several examples.

Before proving that the six functions defined above are bijections we give a data compression meaning of them in the sense that they map some 'regularities' into runs of 0s; and so they can be considered as pre-processing steps for data compression entropy coding (like Huffman or arithmetic coding).

In an integer sequence $t_1 t_2 \ldots t_n$

- a *run* is a length maximal factor $t_i t_{i+1} \ldots t_j$ with $t_i = t_{i+1} = \ldots = t_j$; and
- an *increasing run* is a length maximal factor $t_i t_{i+1} \ldots t_j$ with $t_{k+1} = t_k + 1$ for all k, $i \leq k < j$.

For a subexcedant sequence $s = s_1 s_2 s_3 \ldots s_{n-1} s_n \in S_n$ we define its *complement* s^c as the subexcedant sequence $s_1(1-s_2)(2-s_3)\ldots(n-2-s_{n-1})(n-1-s_n)$; and the complement of a code transform Ξ is the transform $\Xi^c : S_n \to S_n$ with $\Xi^c(s) = (\Xi(s))^c$.

Remark 1.

- *The functions Δ, Γ, Θ and Υ transform each run (except its first or its last element) into a run of* 0*s;*
- *The functions Λ^c and Ψ transform each increasing run (except its first or its last element) into a run of* 0*s.*

Example 1.

- *If $t = 011333 \in S_n$, then $\Delta(t) = 001003$, $\Gamma(t) = 010200$, $\Theta(t) = 010300$, and $\Upsilon(t) = 001003$.*
- *If $t = 012123 \in S_n$, then $\Lambda^c(t) = 000200$, and $\Psi(t) = 002003$.*

Proposition 1. *The six functions given in Definition 2 are bijections.*

Proof. It is enough to prove that each of these functions is injective, cardinality reasons complete the proof. The injectivity of Δ and Γ is routine.

We consider two different sequences $t = t_1t_2\cdots t_n$ and $t' = t'_1t'_2\cdots t'_n$ in S_n.

The injectivity of Θ and Λ. Let i be the leftmost position where t and t' differ. The sequences $s_1s_2\cdots s_n = \Theta(t)$ and $s'_1s'_2\cdots s'_n = \Theta(t')$ differ also in position i. Indeed, if $t_i \le t_{i-1} = t'_{i-1} < t'_i$, then $s_i \le t_{i-1} < s'_i$ and so $s_i \ne s'_i$. The other cases are equivalent or trivial, and the injectivity of Λ is similar.

The injectivity of Υ. Let i be the rightmost position where t and t' differ. The sequences $s_1s_2\cdots s_n = \Upsilon(t)$ and $s'_1s'_2\cdots s'_n = \Upsilon(t')$ differ also in position i. Indeed, if $i < n$ and $t_i \ge t_{i+1} = t'_{i+1} > t'_i$, then $s_i \le i-1-t_{i+1} < s'_i$ and so $s_i \ne s'_i$. The other cases are equivalent or trivial.

The injectivity of Ψ. Let i be the rightmost position where t and t' differ. The sequences $s_1s_2\cdots s_n = \Psi(t)$ and $s'_1s'_2\cdots s'_n = \Psi(t')$ differ also in position i. Indeed, if $i < n$ and $t_i \ge t_{i+1} = t'_{i+1} > t'_i$, then $s_i \ge t_{i+1} > s'_i$ and so $s_i \ne s'_i$. The other cases are equivalent or trivial. $\square$

It is easy to check the following.

Remark 2. *The inverses of Δ, Γ, Θ, Λ, Υ and Ψ are given below. If $s = s_1s_2\cdots s_n \in S_n$, then*

- $\Delta^{-1}(s) = t_1t_2\cdots t_n$ *with* $t_i = \begin{cases} s_n & \text{if} \quad i = n \\ (t_{i+1} + s_i) \mod i & \text{if} \quad 1 \le i < n. \end{cases}$

- $\Gamma^{-1}(s) = t_1t_2\cdots t_n$ *with* $t_i = \begin{cases} s_1 & \text{if} \quad i = 1 \\ (t_{i-1} - s_i) \mod i & \text{if} \quad 1 < i \le n. \end{cases}$

- $\Theta^{-1}(s) = t_1t_2\cdots t_n$ *with* $t_i = \begin{cases} s_1 & \text{if} \quad i = 1 \\ s_i & \text{if} \quad t_{i-1} < s_i \text{ and } 1 < i \le n \\ t_{i-1} - s_i & \text{if} \quad t_{i-1} \ge s_i \text{ and } 1 < i \le n. \end{cases}$

- $\Lambda^{-1}(s) = t_1t_2\cdots t_n$ *with* $t_i = \begin{cases} s_1 & \text{if} \quad i = 1 \\ i + t_{i-1} - s_i & \text{if} \quad t_{i-1} < s_i \text{ and } 1 < i \le n \\ s_i & \text{if} \quad t_{i-1} \ge s_i \text{ and } 1 < i \le n. \end{cases}$

- $\Upsilon^{-1}(s) = t_1t_2\cdots t_n$ *with* $t_i = \begin{cases} s_n & \text{if} \quad i = n \\ t_{i+1} + s_i & \text{if} \quad t_{i+1} + s_i \le i - 1 \text{ and } 1 \le i < n \\ i - 1 - s_i & \text{if} \quad t_{i+1} + s_i > i - 1 \text{ and } 1 \le i < n. \end{cases}$

- $\Psi^{-1}(s) = t_1t_2\cdots t_n$ *with* $t_i = \begin{cases} s_n & \text{if} \quad i = n \\ s_i & \text{if} \quad s_i \ge t_{i+1} \text{ and } 1 \le i < n \\ t_{i+1} - s_i - 1 & \text{if} \quad s_i < t_{i+1} \text{ and } 1 \le i < n. \end{cases}$

The complement of a permutation $\pi = \pi_1\pi_2\pi_3\dots\pi_{n-1}\pi_n \in \mathfrak{S}_n$ is defined as $\pi^c = (n+1-\pi_1)(n+1-\pi_2)(n+1-\pi_3)\dots(n+1-\pi_{n-1})(n+1-\pi_n) \in \mathfrak{S}_n$; and the complement of a subexcedants sequence and of a code transform are defined before Remark 1. It is easy to see that for any $t \in S_n$, $L^{-1}(t^c) = (L^{-1})^c(t)$. In [8] are given some commutative properties of Δ, and the next lemma extends them to the transform Γ.

Lemma 1. *Let Ξ be one of the transforms Δ and Γ. For any $t \in S_n$, we have*

(i) $\Xi(t^c) = (\Xi)^c(t)$,

(ii) $\Xi^{-1}(t^c) = (\Xi^{-1})^c(t)$,

(iii) $(L^{-1} \circ \Xi^{-1})(t^c) = (L^{-1} \circ \Xi^{-1})^c(t)$.

3 Main results

In this section we prove the Mahonicity of some patterns. The results are stated in Theorems 1-5 and summarized in Table 1; all of them are already known [1, 5, 6]. The novelty consists in the unified approach based on permutation codes, described briefly as:

- Find a convenient set of pointed patterns $\{\underline{\sigma}, \underline{\tau}, \dots\}$ corresponding to the set of patterns $\{\sigma, \tau, \dots\}$,
- Show that for all $\pi \in \mathfrak{S}_n$ the mapping $\pi \mapsto t_1t_2\cdots t_n$ with $t_i = (\underline{\sigma} + \underline{\tau} + \cdots)_i\,\pi$, $1 \le i \le n$, is a permutation code based on the Lehmer code of π.

This technique was initiated by the author in [8] where the transform Δ is introduced. Before giving our first theorem, we need some further considerations on this transform. For a permutation $\pi \in \mathfrak{S}_n$ with $L(\pi)$ its Lehmer code, we call $\Delta(L(\pi)) \in S_n$, the *McMahon code* of π. This is justified by the following result which is a consequence of Theorem 13 and Corollary 6 in [8].

Proposition 2. *If $s_1s_2\cdots s_n$ is the McMahon code of $\pi \in \mathfrak{S}_n$, then $\mathsf{MAJ}\,\pi = \sum_{i=1}^{n} s_i$.*

Also in [8] is given the next corollary, expressed here in terms of pattern involvement.

Corollary 1. *For any $\pi \in \mathfrak{S}_n$, the McMahon code $s = s_1s_2\cdots s_n$ of π is given by:*

$$s_i = \begin{cases} ((a\text{-}\underline{c}b) + (b\text{-}\underline{a}c) + (c\text{-}\underline{b}a))_i\,\pi & \text{if} \quad i \neq n \\ (b\text{-}a]\,\pi & \text{if} \quad i = n. \end{cases}$$

A consequence of Corollary 1 is

π	$=$	$5\,2\,1\,6\,4\,3$
$L(\pi)$	$=$	$0\,1\,2\,0\,2\,3$
$\Delta(L(\pi))$	$=$	$0\,1\,2\,2\,4\,3$
$\Gamma(L(\pi))$	$=$	$0\,1\,2\,2\,3\,5$
$\Theta(L(\pi))$	$=$	$0\,1\,2\,2\,2\,3$
$\Lambda(L(\pi))$	$=$	$0\,1\,2\,0\,3\,5$
$\Upsilon(L(\pi))$	$=$	$0\,0\,2\,3\,2\,3$
$\Psi(L(\pi))$	$=$	$0\,0\,2\,1\,0\,3$

(a)

after s_i right circular shifts of the length-i prefix	(i, s_i)
$1\,2\,3\,4\,5\,6$	
$4\,5\,6\,1\,2\,3$	$(6, 3)$
$5\,6\,1\,2\,4\,3$	$(5, 4)$
$1\,2\,5\,6\,4\,3$	$(4, 2)$
$2\,5\,1\,6\,4\,3$	$(3, 2)$
$5\,2\,1\,6\,4\,3$	$(2, 1)$
$5\,2\,1\,6\,4\,3$	$(1, 0)$

(b)

Figure 1: (a) The permutation π together with its Lehmer code and its transforms. (b) The construction of $\pi = 5\,2\,1\,6\,4\,3$ from its McMahon code $s = \Delta(L(\pi)) = 0\,1\,2\,2\,4\,3$.

Corollary 2. *For any $\pi \in \mathfrak{S}_n$ we have*

1. $\mathsf{MAJ}\,\pi = ((a\text{-}cb) + (b\text{-}ac) + (c\text{-}ba) + (b\text{-}a])\,\pi$,
2. *If the McMahon code of π is $s_1 s_2 \cdots s_n$, then*

$$((a\text{-}cb) + (b\text{-}ac) + (c\text{-}ba))\,\pi = \sum_{i=1}^{n-1} s_i.$$

In [8, Theorem 12] is given an algorithmic meaning of the McMahon code $s_1 s_2 \cdots s_n$ of $\pi \in \mathfrak{S}_n$: π can be obtained from the identity $\iota = 1\,2\,\ldots\,n \in \mathfrak{S}_n$ by iteratively performing on ι, s_i right circular shifts of its length-i prefix, for $i = n, n-1, \ldots, 2, 1$. For example, the construction of $\pi = 5\,2\,1\,6\,4\,3 \in \mathfrak{S}_6$ with its McMahon code $0\,1\,2\,2\,4\,3$ is given in Figure 1.b.

With this algorithmic interpretation of McMahon code we have the following remark.

Remark 3. *If $\sigma, \tau \in \mathfrak{S}_n$ are two permutations with their McMahon codes differing only in the last position, then $\sigma_i \neq \tau_i$ for all i, $1 \leq i \leq n$.*

Theorem 1. *The following statistic (statistic S_2 in [1, Conjecture 11]) is Mahonian*

$$(a\text{-}cb) + (b\text{-}ac) + (c\text{-}ba) + [b\text{-}a).$$

Proof. To a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$ with its McMahon code $p_1p_2\cdots p_{n-1}p_n$ we associate a subexcedant sequence $p_1p_2\cdots p_{n-1}(\pi_1-1)$. Clearly, by the second point of Corollary 2

$$((a\text{-}cb) + (b\text{-}ac) + (c\text{-}ba) + [b\text{-}a])\,\pi = \sum_{i=1}^{n-1} p_i + (\pi_1 - 1).$$

Now we show that the mapping

$$\pi \mapsto p_1p_2\cdots p_{n-1}(\pi_1 - 1) \tag{3}$$

is an injection and so (by cardinality reasons) a permutation code.
Let $\sigma \neq \tau$ be two permutations in $\mathfrak{S}_n$ and let i be the leftmost position where $s_1s_2\cdots s_n$ and $t_1t_2\cdots t_n$, their McMahon codes, differ. If $i < n$, then the subexcedant sequences corresponding to σ and τ (defined in the mapping in relation (3)) differ also in position i. If $i = n$, then by Remark 3, $\sigma_1 \neq \tau_1$ and again, the subexcedant sequences corresponding to σ and τ are different. □

The reduction of a sequence of n distinct integers is the permutation in $\mathfrak{S}_n$ obtained by replacing the smallest member by 1, the second-smallest by 2, ..., and the largest by n.

Remark 4. *Let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$. If $\tau \in \mathfrak{S}_{n-1}$ is the reduction of $\pi_2\cdots\pi_n$, then*

$$((a\text{-}cb) + (b\text{-}ca) + (c\text{-}ba))\,\pi \;=\; \mathsf{MAJ}\,\tau.$$

Proof. For all $i \geq 2$, $((a\text{-}\underline{c}b) + (b\text{-}\underline{c}a) + (c\text{-}\underline{b}a))_i\,\pi$ equals $i-1$ if i is a descent in π (and so, if $i-1$ is a descent in τ) and 0 otherwise. Now, summing for all i, $2 \leq i \leq n$, the desired relation holds. □

Theorem 2. *The following statistic (statistic S_4 in [1, Conjecture 11]) is Mahonian*

$$(a\text{-}cb) + (b\text{-}ca) + (c\text{-}ba) + [b\text{-}a].$$

Proof. For $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$ let $\tau \in \mathfrak{S}_{n-1}$ be the reduction of $\pi_2\cdots\pi_n$ and define $s = s_1s_2\cdots s_n \in S_n$ by:

- $s_1s_2\cdots s_{n-1}$ is the McMahon code of τ, and
- $s_n = \pi_1 - 1$.

First, the mapping $\pi \mapsto s$ is a permutation code. Indeed, $s_1s_2\cdots s_{n-1}s_n \in S_n$ and the prefix $s_1s_2\cdots s_{n-1}$ uniquely determines τ, which together with s_n determines π. Now we show that

$$\sum_{i=1}^{n} s_i = ((a\text{-}cb) + (b\text{-}ca) + (c\text{-}ba) + [b\text{-}a])\,\pi. \tag{4}$$

Using Remark 4 we have

$$\begin{aligned}((a\text{-}cb) + (b\text{-}ca) + (c\text{-}ba))\,\pi &= \mathsf{MAJ}\,\tau\\ &= \sum_{i=1}^{n-1} s_i,\end{aligned}$$

and since $s_n = \pi_1 - 1 = [b\text{-}a)\,\pi$ relation (4) holds. $\square$

Theorem 3. *The following statistics (defined in [1, Proposition 9] or equivalent to them) are Mahonian.*

1. $\mathsf{STAT} = (a\text{-}cb) + (b\text{-}ac) + (c\text{-}ba) + (ba)$,
2. $\mathsf{STAT}' = (b\text{-}ac) + (b\text{-}ca) + (c\text{-}ba) + (ba)$,
3. $\mathsf{STAT}'' = (a\text{-}cb) + (c\text{-}ab) + (c\text{-}ba) + (ba)$.

Proof. Let $\pi \in \mathfrak{S}_n$ and t its Lehmer code. We will use the remark that $\pi_{i-1} < \pi_i$ if and only if $t_{i-1} \geq t_i$.
1. Let $s_1 s_2 \cdots s_n = \Gamma(t)$. It is routine to check that

$$(b\text{-}a\underline{c})_i\,\pi = \begin{cases} t_{i-1} - t_i & \text{if} \quad t_{i-1} \geq t_i \\ 0 & \text{if} \quad t_{i-1} < t_i, \end{cases} \tag{5}$$

and

$$((a\text{-}c\underline{b}) + (c\text{-}b\underline{a}) + (b\underline{a}))_i\,\pi = \begin{cases} 0 & \text{if} \quad t_{i-1} \geq t_i \\ i + t_{i-1} - t_i & \text{if} \quad t_{i-1} < t_i, \end{cases} \tag{6}$$

for $1 < i \leq n$.
Summing both relations and using the definition of Γ we have

$$((a\text{-}c\underline{b}) + (b\text{-}a\underline{c}) + (c\text{-}b\underline{a}) + (b\underline{a}))_i\,\pi = s_i,$$

and thus

$$((a\text{-}cb) + (b\text{-}ac) + (c\text{-}ba) + (ba))\,\pi = \sum_{i=1}^{n} s_i.$$

2. Let $s_1 s_2 \cdots s_n = \Theta(t)$. Similarly, we have

$$((b\text{-}c\underline{a}) + (c\text{-}b\underline{a}) + (b\underline{a}))_i\,\pi = \begin{cases} 0 & \text{if} \quad t_{i-1} \geq t_i \\ t_i & \text{if} \quad t_{i-1} < t_i, \end{cases}$$

for $1 < i \leq n$.
Summing this relation with (5) and using the definition of Θ we have

$$((b\text{-}a\underline{c}) + (b\text{-}c\underline{a}) + (c\text{-}b\underline{a}) + (b\underline{a}))_i\,\pi = s_i,$$

and thus

$$((b\text{-}ac) + (b\text{-}ca) + (c\text{-}ba) + (ba))\,\pi = \sum_{i=1}^{n} s_i.$$

3. Now let $s_1 s_2 \cdots s_n = \Lambda(t)$.

$$(c\text{-}a\underline{b})_i\,\pi = \begin{cases} t_i & \text{if} \quad t_{i-1} \geq t_i \\ 0 & \text{if} \quad t_{i-1} < t_i, \end{cases}$$

for $1 < i \leq n$, which together with (6) and the definition of Λ gives

$$((a\text{-}c\underline{b}) + (c\text{-}a\underline{b}) + (c\text{-}b\underline{a}) + (b\underline{a}))_i\,\pi = s_i,$$

and so

$$((a\text{-}cb) + (c\text{-}ab) + (c\text{-}ba) + (ba))\,\pi = \sum_{i=1}^{n} s_i.$$

Since Γ, Θ and Λ are code transforms, it follows that the three statistics are Mahonian. □

Before proving our next theorem we need the following result.

Lemma 2. *For any integer n and permutation $\pi \in \mathfrak{S}_n$ we have*

$$((b\text{-}ca) + (ba))\,\pi = ((b\text{-}ac) + (b\text{-}a])\,\pi.$$

Proof. We express $\mathsf{MAJ}\,\pi$ in to ways:

$$\begin{array}{rcll} \mathsf{MAJ}\,\pi & = & ((a\text{-}cb) + (b\text{-}ca) + (c\text{-}ba) + (ba))\,\pi & \text{by definition} \\ & = & ((a\text{-}cb) + (b\text{-}ac) + (c\text{-}ba) + (b\text{-}a])\,\pi & \text{by Corollary 2.1} \end{array}$$

and the result holds. □

Theorem 4. *The following statistic (second statistic in [1, Conjecture 8]) is Mahonian*

$$(a\text{-}cb) + (b\text{-}ca) + (b\text{-}ca) + (ba).$$

Proof. Let $\pi \in \mathfrak{S}_n$ with its Lehmer code $t = t_1 t_2 \cdots t_n$ and $s = s_1 s_2 \cdots s_n = \Upsilon(t)$. As previously, it is easy to check that

$$((a\text{-}\underline{c}b) + (b\text{-}\underline{c}a))_i\,\pi = \begin{cases} i - t_i - 1 & \text{if} \quad t_i < t_{i+1} \\ 0 & \text{if} \quad t_i \geq t_{i+1}, \end{cases}$$

and

$$(b\text{-}\underline{a}c)_i\,\pi = \begin{cases} 0 & \text{if} \quad t_i < t_{i+1} \\ t_i - t_{i+1} & \text{if} \quad t_i \geq t_{i+1}, \end{cases}$$

for $1 \leq i < n$.

Summing both relations and using the definition of Υ we have

$$s_i = \begin{cases} ((a\text{-}\underline{c}b) + (b\text{-}\underline{a}c) + (b\text{-}\underline{c}a))_i\,\pi & \text{if} \quad i \neq n \\ (b\text{-}a]\,\pi & \text{if} \quad i = n. \end{cases}$$

Now, by Lemma 2 and the previous relation we have

$$\begin{aligned} ((a\text{-}cb) + (b\text{-}ca) + (b\text{-}ca) + (ba))\,\pi &= ((a\text{-}cb) + (b\text{-}ac) + (b\text{-}ca) + (b\text{-}a])\,\pi \\ &= \sum_{i=1}^{n} s_i, \end{aligned}$$

and since Υ is a code transform the result holds. $\square$

Theorem 5. *The following statistic (equivalent with the third one in [1, Conjecture 8]) is Mahonian*

$$(b\text{-}ca) + (b\text{-}ca) + (c\text{-}ab) + (ba).$$

Proof. Let $\pi \in \mathfrak{S}_n$ with its Lehmer code $t = t_1 t_2 \cdots t_n$ and $s = s_1 s_2 \cdots s_n = \Psi(t)$. We have

$$((b\text{-}\underline{a}c) + (c\text{-}\underline{a}b))_i\,\pi = \begin{cases} 0 & \text{if} \quad t_i < t_{i+1} \\ t_i & \text{if} \quad t_i \geq t_{i+1}, \end{cases}$$

and

$$(b\text{-}\underline{c}a)_i\,\pi = \begin{cases} t_{i+1} - t_i - 1 & \text{if} \quad t_i < t_{i+1} \\ 0 & \text{if} \quad t_i \geq t_{i+1}, \end{cases}$$

for $1 \leq i < n$, which together with the definition of Ψ gives

$$s_i = \begin{cases} ((b\text{-}\underline{a}c) + (b\text{-}\underline{c}a) + (c\text{-}\underline{a}b))_i\,\pi & \text{if} \quad i \neq n \\ (b\text{-}a]\,\pi & \text{if} \quad i = n. \end{cases}$$

Now, by Lemma 2 and the previous relation we have

$$\begin{aligned} ((b\text{-}ca) + (b\text{-}ca) + (c\text{-}ab) + (ba))\,\pi &= ((b\text{-}ac) + (b\text{-}ca) + (c\text{-}ab) + (b\text{-}a])\,\pi \\ &= \sum_{i=1}^{n} s_i, \end{aligned}$$

and since Ψ is a code transform the result holds. $\square$

statistic		transform of Lehmer code (if any)
INV	$(b\text{-}a)$ $(b\text{-}ca) + (c\text{-}ab) + (c\text{-}ba) + (ba)$	
MAJ (Proposition 2)	$(a\text{-}cb) + (b\text{-}ca) + (c\text{-}ba) + (ba)$ $(a\text{-}cb) + (b\text{-}ac) + (c\text{-}ba) + (b\text{-}a]$	Δ
Theorem 1 (statistic S_2 in [1, Conjecture 11])	$(a\text{-}cb) + (b\text{-}ac) + (c\text{-}ba) + [b\text{-}a)$	
Theorem 2 (statistic S_4 in [1, Conjecture 11])	$(a\text{-}cb) + (b\text{-}ca) + (c\text{-}ba) + [b\text{-}a)$	
Theorem 3.1 (STAT in [1, Proposition 9])	$(a\text{-}cb) + (b\text{-}ac) + (c\text{-}ba) + (ba)$	Γ
Theorem 3.2 (STAT$'$ in [1, Proposition 9])	$(b\text{-}ac) + (b\text{-}ca) + (c\text{-}ba) + (ba)$	Θ
Theorem 3.3 (STAT$''$ in [1, Proposition 9])	$(a\text{-}cb) + (c\text{-}ab) + (c\text{-}ba) + (ba)$	Λ
Theorem 4 (second statistic in [1, Conjecture 8])	$(a\text{-}cb) + (b\text{-}ca) + (b\text{-}ca) + (ba)$	Υ
Theorem 5 (third statistic in [1, Conjecture 8])	$(b\text{-}ca) + (b\text{-}ca) + (c\text{-}ab) + (ba)$	Ψ

Table 1: Pattern statistics together with their Lehmer code transforms.

4 Final remarks

The six code transforms given in Definition 2 provide constructive bijections between permutations having a given value for some Mahonian statistics. For example, the mapping

$$\{\pi \in \mathfrak{S}_n \,|\, \mathsf{STAT}\,\pi = k\} \to \{\pi \in \mathfrak{S}_n \,|\, \mathsf{STAT}'\,\pi = k\}$$

defined by

$$\pi \mapsto L^{-1} \circ \Theta^{-1} \circ \Gamma \circ L(\pi)$$

gives a constructive bijection for $0 \leq k \leq \binom{n}{2}$, where L denotes the Lehmer code.

Each statistic giving the number of occurrences of a pattern of the form $(*\text{-}**)$ is a linear combination of other statistics. Indeed, from Table 1 it is routine to check for example, the following expressions, where des is the Eulerian statistic which gives the number of descents in a permutation, *i.e.*, $\mathsf{des}\,\pi = (ba)\,\pi$; and υ denotes the Mahonian statistic corresponding to the transform Γ, *i.e.*, $\upsilon\,\pi = \sum_{i=1}^{n} s_i$, with $s_1 s_2 \cdots s_n = \Gamma(t)$, and t the Lehmer code of π.

$$(a\text{-}cb)\,\pi = \tfrac{1}{3} \cdot (\upsilon + 2 \cdot \mathsf{STAT}'' - 2 \cdot \mathsf{INV}\,\pi - \mathsf{des})\,\pi,$$

$$(b\text{-}ac)\,\pi = (\mathsf{STAT}' - \mathsf{MAJ})\,\pi + \tfrac{1}{3} \cdot (\upsilon + 2 \cdot \mathsf{STAT}'' - 2 \cdot \mathsf{INV} - \mathsf{des})\,\pi,$$

$$(b\text{-}ca)\,\pi = \tfrac{1}{3} \cdot (\upsilon + \mathsf{INV} - \mathsf{STAT}'' - \mathsf{des})\,\pi,$$

$$(c\text{-}ab)\,\pi = \tfrac{1}{3} \cdot (\upsilon + \mathsf{INV} + 2 \cdot \mathsf{STAT}'' - \mathsf{des})\,\pi - \mathsf{MAJ}\,\pi,$$

$$(c\text{-}ba)\,\pi = \mathsf{MAJ}\,\pi + \tfrac{1}{3} \cdot (\mathsf{INV} - 2 \cdot \upsilon - \mathsf{STAT}'' - \mathsf{des})\,\pi, \text{ and}$$

$$(a\text{-}bc)\,\pi = \tfrac{(n-2)\cdot(n-1)}{2} - x, \text{ with } x \text{ the sum of the previous five statistics.}$$

Note that each of these single pattern statistic can be computed in $O(n)$ time from the Lehmer code t of $\pi \in \mathfrak{S}_n$, and t in turn can be computed in $O(n \log n)$ time by an adaptation of an appropriate sorting algorithm.
Also, the Mahonian statistics in Table 1 are linearly dependant and we have, for example:

$$\mathsf{STAT}\,\pi = \mathsf{STAT}'\,\pi + \mathsf{STAT}''\,\pi - \mathsf{INV}\,\pi, \text{ and}$$

$$S_4\,\pi = S_2\,\pi + \mathsf{STAT}\,\pi - \mathsf{MAJ}\,\pi = (\mathsf{STAT} - \mathsf{des})\,\pi - \pi_1 - 1,$$

and similar relations can be expressed for other statistics.

We conclude this paper with a few open questions.

- Do the bijections mentioned in the beginning of this section preserve any other statistic?

- Do they restrict to some pattern avoiding classes?

- In [9], based on the Lehmer code for permutations, T. Walsh gives an exhaustive generating algorithm for permutations with a fixed value for the statistic INV (i.e., with a fixed number of inversions). Can our code transforms be used for generating permutations with a given value for other Mahonian statistics?

Acknowledgements

The author wishes to thank the anonymous referees, whose remarks and suggestions improved the overall quality of the paper.

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