Some Generalizations of a Simion-Schmidt Bijection

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Abstract

In 1985 Simion and Schmidt gave a constructive bijection φ from the set of all length $(n-1)$ binary strings having no two consecutive 1s to the set of all length n permutations avoiding all patterns in $\{123, 132, 213\}$.

In this paper we generalize φ to an injective function from $\{0,1\}^{n-1}$ to the set S_n of all length n permutations and derive from it four bijections $\varphi : P \to Q$ where $P \subseteq \{0,1\}^{n-1}$ and $Q \subset S_n$. The domains are sets of restricted binary strings and the codomains are sets of pattern-avoiding permutations. As a particular case we retrieve the original Simion-Schmidt bijection.

We also show that the bijections obtained are actually combinatorial isomorphisms, i.e., closeness-preserving bijections. Three of them have known Gray codes and generating algorithms for their domains and we present similar results for each corresponding codomain, under the appropriate combinatorial isomorphism.
Keywords: pattern-avoiding permutations, Fibonacci and Lucas strings, constructive bijections, combinatorial isomorphisms, Gray codes.

1 Introduction and Motivation

A permutation π of the set of integers $[n] = \{1, 2, \ldots, n\}$ is a bijection from $[n]$ onto itself and we denote by S_n the set of all such permutations. For two permutations $\tau \in S_k$ and $\pi \in S_n$, with $k < n$, we say that a subsequence $\pi_{\ell_1}, \pi_{\ell_1}, \ldots, \pi_{\ell_k}$ of π is of *type* τ whenever $\pi_{\ell_i} < \pi_{\ell_j}$ if and only if $\tau_i < \tau_j$ for all i, j, $1 \le i, j \le k$. In this context the permutation τ is called *pattern*. For example, the subsequence 523 of the permutation 15423 is of type 312. Now let $T = \{\tau_1, \tau_2, \ldots, \tau_k\}$ be a set of patterns. We say π *avoids* T whenever π contains no subsequence of type τ_i for all $\tau_i \in T$, and we will denote by $S_n(T)$ the set of all such permutations. For example, the permutation $15423 \in S_5$ avoids the set of patterns $\{231, 213\}$ because it has no subsequence of type 231 or 213; so we have $15423 \in S_5(231, 213)$ but $15423 \notin S_5(312)$. Clearly $S_n(T_1) \subset S_n(T_2)$ if $T_2 \subset T_1$.

This paper is inspired by Simion-Schmidt's result [1, Proposition 15*] which gave a constructive bijection φ from F_{n-1} to $S_n(123, 132, 213)$. Here F_{n-1} is the set of all length $(n-1)$ binary strings having no two consecutive 1s; such strings are called *Fibonacci strings*. In this paper we generalize φ to an injective but not surjective function φ from $\{0,1\}^{n-1}$ to S_n and we derive four bijections $\varphi : P \to Q$ where $P \subseteq \{0,1\}^{n-1}$ and $Q \subset S_n$. The pairs (P, Q) are:

1. $\big(\{0,1\}^{n-1}, S_n(123, 132)\big)$,

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2. $\left(F^{(p)}_{n-1}, S_n(123, 132, \sigma_p)\right)$, where $F^{(p)}_{n-1}$ is the set of length $(n-1)$ binary strings with no p consecutive ones and σ_p is the length $(p+1)$ permutation $p(p-1)(p-2)\dots 1(p+1)$ (when $p=2$, φ becomes the original Simion-Schmidt bijection),
3. $\left(C_{n-1,k}, S'_{n,k}(123, 132)\right)$, where $C_{n-1,k}$ is the set of binary strings in $\{0,1\}^{n-1}$ having exactly k 1s and $S'_{n,k}(123, 132)$ is the set of permutations in $S_n(123, 132)$ having exactly k *non-inversions* (see Definition 1), and
4. $\left(K^{(p)}_{n-1}, S_n(123, 132, \sigma_p, T_p)\right)$, where $K^{(p)}_{n-1}$ is a class of binary strings counted by the Lucas number and T_p is a set of *generalized patterns* (defined formally after the proof of Lemma 3).

We denote by φ the Simion-Schmidt bijection and its extensions. We prove that the four bijections $\varphi : P \to Q$ are combinatorial isomorphisms; that is, closeness-preserving bijections. The first three of them have known Gray codes and generating algorithms for their domains; hence we present a Gray code and sketch a generating algorithm for $S_n(123, 132)$, $S_n(123, 132, \sigma_p)$, and $S'_{n,k}(123, 132)$, which are the images of these domains under the corresponding combinatorial isomorphism.

The interest of our results is two-fold: we show that the Simion-Schmidt bijection can be extended to other combinatorial objects and all of those bijections are isomorphisms, that is, satisfy strong combinatorial properties and so make it possible to transform, under these isomorphisms, some known properties of certain binary strings into the corresponding properties of permutations avoiding certain patterns. This paper is the extended version of [2] and the rest of it is organized as follows. Section 2 presents the generalization of the Simion-Schmidt bijection and the derivation of the four bijections. Section 3 shows that these bijections are actually combinatorial isomorphisms, Section 4 proposes a Gray code for each of the codomains of the first three bijections, and Section 5 presents some graph-theoretical consequences and sketches a generating algorithm for each Gray code. The final section gives some concluding remarks.

2 The Generalized Simion-Schmidt Injection

For any $b = b_1 b_2 \cdots b_{n-1} \in \{0,1\}^{n-1}$ we construct a permutation $\pi \in S_n$ which has its i-th entry, π_i, given by the following rule. If $X_i = \{1, 2, \cdots, n\} - \{\pi_1, \pi_2, \cdots, \pi_{i-1}\}$, then set

$$\pi_i = \begin{cases} \text{the largest element in } X_i \text{ if } b_i = 0 \\ \text{the second largest element in } X_i \text{ if } b_i = 1 \end{cases} \tag{1}$$

and finally π_n is the single element in X_n.

We denote by $\varphi(b)$ the unique image of $b \in \{0,1\}^{n-1}$ under this procedure. Furthermore, two different strings in $\{0,1\}^{n-1}$ are mapped into two different permutations in S_n; therefore $\varphi : \{0,1\}^{n-1} \to S_n$ is an injective function and cardinality considerations show that it is not a bijection.

The construction above was already given by Simion and Schmidt [1] in a particular context, namely as a bijection between length $(n-1)$ binary strings with no two consecutive ones and permutations in $S_n(123, 132, 213)$. In this section we generalize their result.

The cases $\varphi : \{0,1\}^{n-1} \to S_n(123,132)$ and $\varphi : F^{(p)}_{n-1} \to S_n(123,132,\sigma_p)$

The enumeration of permutations avoiding the patterns 123 and 132 appears for instance in [3] and they characterization in [4] and later in [5]; in [4] are also enumerated permutations avoiding two length 3 patterns and a length p pattern. The next lemma gives two generalizations of φ; instead of using the characterization in [4], we give here, for the sake of clarity, a complete proof.

Lemma 1.

1. *$\varphi : \{0,1\}^{n-1} \to S_n(123,132)$ is a bijection.*
2. *$\varphi : F^{(p)}_{n-1} \to S_n(123,132,\sigma_p)$ is a bijection, where $F^{(p)}_{n-1}$ is the set of length $(n-1)$ binary strings having no p consecutive 1s and σ_p is the length $(p+1)$ permutation $p(p-1)(p-2)\dots 1(p+1)$.*

Proof. 1. Suppose that $\pi \in S_n(123,132)$ and that k, $1 \le k \le n$, are such that $\pi_k = n$. If $k > 1$, then $\pi_i > \pi_{i+1}$ for all i, $1 \le i < k-1$, otherwise 123 could not be avoided. Moreover, $\pi_i > \pi_j$ for all $i < k$ and $j > k$, otherwise $\pi_i\pi_k\pi_j$ is a sequence of type 132. Therefore, $\pi = \pi_1\pi_2\dots\pi_k\pi'$ with $\pi_i = \pi_{i+1}+1$ for $1 \le i < k-1$ and $\pi' \in S_{n-k}(123,132)$.

If $\pi \in S_n(123,132)$, then by induction, there exist integers $0 = k_0 < k_1 < \dots < k_r < \dots < k_m = n$ such that π is a sequence of m blocks

$$\pi = \underbrace{\pi_1\pi_2\dots\pi_{k_1}}\dots\underbrace{\pi_{k_{r-1}+1}\pi_{k_{r-1}+2}\dots\pi_{k_r}}\dots\underbrace{\pi_{k_{m-1}+1}\pi_{k_{m-1}+2}\dots\pi_{k_m}} \tag{2}$$

such that

- the rightmost elements of each block are in decreasing order: $n = \pi_{k_1} > \pi_{k_2} > \dots > \pi_{k_m}$, and
- in each block containing more than one element
  - the first element equals the last one minus one: $\pi_{k_{r-1}+1} = \pi_{k_r} - 1$, and
  - all elements, except the last one, are consecutive integers in decreasing order: $\pi_\ell = \pi_{\ell+1}+1$ for $k_{r-1}+1 \le \ell < k_r - 1$.

It is easy to check that $b \in \{0,1\}^{n-1}$ defined by

$$b_i = \begin{cases} 0 \text{ if } i = k_r \text{ for some } r, 1 \le r < m \\ 1 \text{ otherwise} \end{cases}$$

satisfies $\varphi(b) = \pi$.
2. In addition, if π avoids $\sigma_p = p(p-1)(p-2)\dots 1(p+1)$, then each block $\pi_{k_{r-1}+1}\pi_{k_{r-1}+2}\dots\pi_{k_r}$ has length at most p and b as defined above has no p consecutive 1s. $\square$

Figure 1 shows the permutations $\underline{9}7654\underline{8}21\underline{3}$ and $\underline{9}7\underline{8}54\underline{6}21\underline{3}$ in $S_9(123,132)$ in array representation; the rightmost element of each block, as mentioned in the proof above, is underlined. Table 1 gives the domains and codomains of the bijection $\varphi : \{0,1\}^{n-1} \to S_n(123,132)$, and $\varphi : F^{(2)}_{n-1} \to S_n(123,132,213)$, for $n = 5$. The listing actually is in Gray code order (see Section 4).

Table 1: (a) The set $\{0,1\}^4$ and $S_5(123,132)$, its image under φ.
(b) The set $F_4^{(2)}$ and $S_5(123,132,213)$, its image under φ.

(a)

$\{0,1\}^4$	$S_5(123,132)$
0111	53214
0110	53241
0100	53421
0101	53412
0001	54312
0000	54321
0010	54231
0011	54213
1011	45213
1010	45231
1000	45321
1001	45312
1101	43512
1100	43521
1110	43251
1111	43215

(b)

$F_4^{(2)}$	$S_5(123,132,213)$
0100	53421
0101	53412
0001	54312
0000	54321
0010	54231
1010	45231
1000	45321
1001	45312

The case $\varphi : C_{n-1,k} \to S'_{n,k}(123,132)$

Now let $C_{n-1,k}$ be the set of strings in $\{0,1\}^{n-1}$ with exactly k occurrences of 1. Strings in $C_{n-1,k}$ are the usual binary string representation of the combinations of k objects chosen from $(n-1)$; so $|C_{n-1,k}| = \binom{n-1}{k}$.

Definition 1. *In a permutation $\pi \in S_n$ a pair (i,j), with $i<j$, is called an* inversion *iff $\pi_i > \pi_j$ and a* non-inversion *iff $\pi_i < \pi_j$.*

We write $S'_{n,k}(T)$ to denote the set of permutations in $S_n(T)$ having exactly k non-inversions.

Lemma 2. $\varphi : C_{n-1,k} \to S'_{n,k}(123,132)$ *is a bijection.*

Proof. Suppose that $b \in \{0,1\}^{n-1}$ and that i, $1 \le i < n$, are such that $b_i = 1$. Then i induces exactly one non-inversion in $\pi = \varphi(b)$. Indeed, let j be the position of the leftmost 0 bit in b at

Table 2: The set $C_{4,2}$ of all length 4 binary strings with 2 occurrences of 1s and $S'_{5,2}(123,132)$ its image under φ.

$C_{4,2}$	$S'_{5,2}(123,132)$
0110	53241
0101	53412
0011	54213
1010	45231
1001	45312
1100	43521

the right of i if any, and $j = n$ otherwise. In π, $\pi_i > \pi_\ell$ for all $\ell > i$ except for $\ell = j$; so (i, j) is a non-inversion and the number of non-inversions in π equals the number of 1s in b. $\square$

Table 2 shows the domain and codomain of the bijection $\varphi : C_{n-1,k} \to S'_{n,k}(123, 132)$ for $n = 5$ and $k = 2$. $C_{4,2}$ is listed so that consecutive strings differ in two positions; when these positions are consecutive the corresponding permutations differ in 3 positions and in 4 positions otherwise. As we will see in Section 3 this is valid for all n.

By considering the transformation given by (1) of the function φ it is easy to check the following.

Remark 1. *Let i, $1 \leq i \leq n-1$, and $b, b' \in \{0,1\}^{n-1}$ and suppose that $b_\ell = b'_\ell$ except for $\ell = i$. Then $\pi = \varphi(b)$ and $\pi' = \varphi(b')$ are such that $\pi_\ell = \pi'_\ell$ except for $\ell \in \{i, j\}$ with j being as follows: the leftmost position at the right of i where $b_j = 0$ if any, and n otherwise.*

The case $\varphi : K^{(p)}_{n-1} \to S_n(123, 132, \sigma_p, T_p)$

We now consider the set $K^{(p)}_n$ of binary strings in $F^{(p)}_n$ which contain at least two 0s in their length $p+1$ prefix. For example, the only binary string in $F^{(2)}_4$ but not in $K^{(2)}_4$ is 1010 (see Table 1) and $F^{(3)}_5 \setminus K^{(3)}_5 = \{10110, 11011, 11010\}$. In the following we show that, in general, the cardinality of $K^{(p)}_n$ is given by the p-th order Lucas number [6] and the image of $K^{(p)}_n$ under φ is a class of (generalized) pattern-avoiding permutations.

Lemma 3. $|K^{(p)}_n| = |F^{(p)}_n|$ *if $n \leq p$ and $|K^{(p)}_n|$ is the n*-th*, p*-th *order Lucas number otherwise.*

Proof. Obviously, for $n \leq p$, any string in $F^{(p)}_n$ is also in $K^{(p)}_n$. For the second part of the statement, we give an indirect proof by showing that $K^{(p)}_n$, $n > p$, is equinumerous with a set of binary strings which in turn is known to be counted by the p-*th* order Lucas number. Consider the set $L^{(p)}_n$ of binary strings in $F^{(p)}_n$ which do not begin with 1^u and end with 1^v such that $u + v \geq p$. $L^{(p)}_n$ is extensively studied in the literature and it is known as the set of Lucas strings since $|L^{(p)}_n|$ is the n-th Lucas number of p-th order (see for instance [6]). Clearly, each string in $L^{(p)}_n$, $n > p$, contains at least two 0s; so it has the form $1^u 0 f 0 1^v$ with $u + v < p$ and $f \in F^{(p)}_k$ for an appropriate k; in addition, the string $1^u 0 1^v 0 f$ belongs to $K^{(p)}_n$. It is easy to see that for any $\ell \in L^{(p)}_n$, $n > p$, with ℓ of the form $1^u 0 f 0 1^v$, the transformation $\ell \to 1^u 0 1^v 0 f$ induces a bijection from $L^{(p)}_n$ to $K^{(p)}_n$. $\square$

In [7] a class of pattern-avoiding permutations counted by Lucas numbers is presented. The patterns are ':' and '$^-$' generalized patterns and the result is obtained via the generating trees method. The next lemma extends φ to this class.

For two permutations $\tau \in S_k$ and $\pi \in S_n$ with $k \leq n$ and an integer j, $1 \leq j < k$, we say that a subsequence $\pi_{\ell_1}\pi_{\ell_2}\ldots\pi_{\ell_k}$ of π is of *generalized type* $\tau_1\tau_2\ldots\tau_j : \tau_{j+1}\ldots\tau_{k-1}\tau_k$ whenever $\pi_{\ell_1}, \pi_{\ell_2}, \ldots, \pi_{\ell_k}$ is of type τ and $\ell_{j+1} = \ell_j + 1$. π avoids the *generalized pattern* $\tau_1\tau_2\ldots\tau_j{:}\tau_{j+1}\ldots\tau_{k-1}\tau_k$ if it avoids the classical pattern $\tau_1\tau_2\ldots\tau_j\tau_{j+1}\ldots\tau_{k-1}\tau_k$ and the symbols which play the role of τ_j and τ_{j+1} in π are adjacent. For example, $564231 \in S_6(34{:}12)$ but 645231 and 563421 are not in $S_6(34{:}12)$.

A barred pattern $\overline{\tau}$ of length k is a (possibly generalized) pattern having a bar over one of its elements. Let τ be the length k pattern identical to $\overline{\tau}$ but unbarred and $\hat{\tau}$ the pattern on $\{1, 2, \ldots, k-1\}$ made up of the $(k-1)$ unbarred elements of $\overline{\tau}$. A permutation $\pi \in S_n$ avoids the pattern $\overline{\tau}$ if any subsequence of type $\hat{\tau}$ in π can be extended to a sequence of type τ. For example $645231 \in S_6(\overline{5}34{:}12)$ and $563421 \notin S_6(\overline{5}34{:}12)$.

For a given $p \geq 2$ and k, $2 \leq k \leq p$, let u_k denote the length $p-k+1$ decreasing sequence $(p+1)p\ldots(k+1)$ and v_k the length $k-1$ decreasing sequence $(k-1)(k-2)\ldots 1$. We now consider the set of generalized patterns

$$T_p = \{\overline{(p+3)}u_k(p+2){:}v_k k \mid 2 \leq k \leq p\}.$$

For example, $T_2 = \{\overline{5}34{:}12\}$ and $T_3 = \{\overline{6}435{:}12, \overline{6}45{:}213\}$.

We note that the permutations in $S_n(123, 132, \sigma_p, T_p)$ are precisely those in $S_n(123, 132, \sigma_p)$, where the total length of the two initial blocks (as in the proof of Lemma 1) does not exceed $p+1$. Since $K^{(p)}_{n-1}$ is the set of strings in $F^{(p)}_{n-1}$ with at least two 0s in their $p+1$ prefix we can state:

Lemma 4. *$\varphi : K^{(p)}_{n-1} \to S_n(123, 132, \sigma_p, T_p)$ is a bijection.*

3 The Isomorphism φ

In a combinatorial class we say that two objects are *close* if they differ in some pre-specified, usually small, way; the Hamming distance is a customary specification, see for instance [8]. A (*combinatorial*) *isomorphism* between two combinatorial classes is a *closeness-preserving* bijection, i.e., two objects in a class are close if and only if their images under this bijection are also close. In this section we show that the bijections in Lemmata 1, 2 and 4 are actually isomorphisms.

Definition 2.

1. *Two binary strings in $\{0,1\}^{n-1}$ are* 1-close *if they differ in a single position.*
2. *Two permutations in $S_n(123, 132)$ are* 1-close *if they differ by the transposition of two entries.*

For example, the binary strings 0111 and 0110 are 1-close and so are their images under φ, i.e., the permutations 53214 and 53241.

Lemma 5. *Let $b, b' \in \{0,1\}^{n-1}$ and $\pi = \varphi(b), \pi' = \varphi(b') \in S_n(123, 132)$. The following propositions are equivalent:*

1. *b and b' are 1-close in $\{0,1\}^{n-1}$,*
2. *π and π' are 1-close in $S_n(123, 132)$,*
3. *the decomposition in blocks of π' (as in relation (2)) is obtained from the one of π either by splitting a block (into two adjacent blocks) or by merging two adjacent blocks.*

Proof. '1 $\Rightarrow$ 2'. This implication follows directly from Remark 1.
'2 $\Rightarrow$ 3'. Let $\pi \in S_n(123, 132)$ and suppose π' is obtained from π by transposing the entries in positions i and j, $1 \leq i < j \leq n$. If π' avoids 123 and 132 then, in the permutation π (with the notations in relation (2)), j must be the rightmost entry in its block and i is either (a) in the same block as j, or (b) the rightmost entry of the precedent block. In case (a) π' is obtained from

π by splitting the block containing j into two blocks and in case (b) by merging two adjacent blocks.
'$3 \Rightarrow 1$'. By considering the definition of the function φ, with the notations in the previous point, we find that $b_\ell = b'_\ell$ except for $\ell = i$. In case (a) $b_i = 1$ and $b'_i = 0$; and in case (b) $b_i = 0$ and $b'_i = 1$. See Figure 1 for an example. □

By the above lemma and since the restriction of a combinatorial isomorphism to one of its subclasses remains a combinatorial isomorphism we have:

Corollary 1. *The bijections*

- $\varphi : \{0,1\}^{n-1} \to S_n(123, 132)$,
- $\varphi : F^{(p)}_{n-1} \to S_n(123, 132, \sigma_p)$ *and*
- $\varphi : K^{(p)}_{n-1} \to S_n(123, 132, \sigma_p, T_p)$

are combinatorial isomorphisms.

Under Definition 2, $C_{n-1,k}$ contains no 1-close strings and now we relax this definition.

Definition 3.

1. *Two binary strings in* $C_{n-1,k}$ *are* 2-close *if they differ by the transposition of two bits.*
2. *Two permutations in* $S'_{n,k}(123, 132)$ *are* 2-close *if they differ by two transpositions.*

Corollary 2. *The bijection* $\varphi : C_{n-1,k} \to S'_{n,k}(123, 132)$ *is a combinatorial isomorphism under Definition 3.*

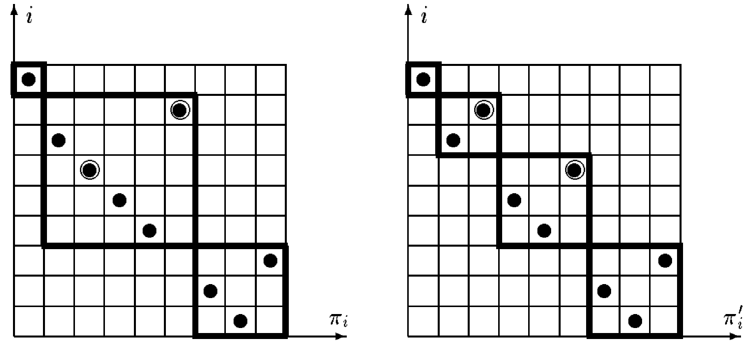
Proof. Let b and b' be two 2-close strings in $C_{n-1,k}$, and π and π' their images in $S'_{n,k}(123, 132)$ under the bijection φ. Consider a binary string $c \in \{0,1\}^{n-1}$ such that c differs from b and from b' in a single position and $\delta \in S_n(123, 132)$ the image of c under φ. Notice that $c \notin C_{n-1,k}$; so $\delta \notin S'_{n,k}(123, 132)$, and there are two such strings c. By Lemma 5, δ differs from π and from π' by a transposition, thus π differs from π' by two transpositions. Similarly, if π and π' are 2-close in $S'_{n,k}(123, 132)$, then so are their pre-images in $C_{n-1,k}$. □

Notice that when the two transpositions in the previous proof have no disjoint domains, π and π' differ by a length three cycle. For example, the transition from the first permutation to the second one in $S'_{5,2}(123, 132)$ as shown in Table 2, namely from 53241 to 53412, is done via a length three cycle.

4 Gray Codes

A list $\mathcal{L}$ for a string set L is an ordered list of the elements of L. If the elements of $\mathcal{L}$ are in some order such that two consecutive elements are close, the list is called a *Gray code list.*

Let α be an integer or a string and $\mathcal{L}$ a list of strings. Then $\alpha \cdot \mathcal{L}$ denotes the list obtained by concatenating α to each string of $\mathcal{L}$; e.g., if $\alpha = 4$ and $\mathcal{L} = \{123, 132, 213\}$, then $\alpha \cdot \mathcal{L} = \{4123, 4132, 4213\}$. If $\mathcal{L}'$ is another list, then $\mathcal{L} \circ \mathcal{L}'$ is the concatenation of the two lists; e.g., if $\mathcal{L}' = \{231, 312, 321\}$, then $\mathcal{L} \circ \mathcal{L}' = \{123, 132, 213, 231, 312, 321\}$. Furthermore, by $\overline{\mathcal{L}}$ we denote



(a) (b)

Figure 1: The permutations 976548213 and 978546213 in $S_9(123,132)$ in array representation. Transposing two entry results in block-splitting (from (a) to (b)), or block-merging (from (b) to (a)).

the reverse of the list $\mathcal{L}$ and $\mathcal{L}^*$ is the list obtained from $\mathcal{L}$ by increasing the largest entry in all strings of $\mathcal{L}$ by 1. Thus, with $\mathcal{L}$ as above, $\overline{\mathcal{L}} = \{213, 132, 123\}$ and $\mathcal{L}^* = \{124, 142, 214\}$.

In this section we construct Gray codes for $S_n(123,132)$, $S_n(123,132,\sigma_p)$, and $S'_{n,k}(123,132)$ from Gray codes for their pre-images under the bijection φ. We begin this section with the concept of *dual reflected order*. The dual reflected order, defined below, is a slight modification of reflected order [9] and, like lexicographical order, both of them are particular cases of *genlex order* [10], that is, any set of strings listed in such an order has the property that strings with a common prefix are contiguous.

Definition 4 ([11]). *For two strings $b = b_1b_2\ldots b_n$ and $b' = b'_1b'_2\ldots b'_n$ in $\{0,1\}^n$ we say that b is less than b' in* dual reflected order *if $b_1b_2\ldots b_k$, the length k prefix of b, contains an odd number of 0s, where k is the leftmost position with $b_k \neq b'_k$.*

In [11] it is noted that: (1) b is less than b' in dual reflected order iff $\overline{b'}$ is less than $\overline{b}$ in reflected order, with $\overline{b}$ and $\overline{b'}$ the bitwise complement of b and b'; (2) like reflected order, dual reflected order induces a Gray code for $\{0,1\}^n$ and $C_{n,k}$, but only the last one yields a Gray code for $F_n^{(p)}$. Here we adopt this order relation in constructing Gray codes for $S_n(123,132)$, $S_n(123,132,213)$, and $S'_{n,k}(123,132)$.

Gray code for $S_n(123,132)$

The following Gray code for the set $\{0,1\}^n$ can be obtained from the famous Binary Reflected Gray Code [9] by replacing in it all 0 bits in each string by 1 bits and vice-versa, and then reversing the obtained list; two consecutive strings differ in a single position and the listing order is dual reflected order [12].

$$\mathcal{B}_n = \begin{cases} \emptyset & \text{if} \quad n = 0 \\ 0 \cdot \overline{\mathcal{B}}_{n-1} \circ 1 \cdot \mathcal{B}_{n-1} & \text{if} \quad n \geq 1. \end{cases} \tag{3}$$

By applying φ, the list $\mathcal{B}_{n-1}$ is transformed into the following list for the set $S_n(123,132)$:

$$\mathcal{S}_n(123,132) = \begin{cases} \{1\} & \text{if} \quad n = 1 \\ n \cdot \overline{\mathcal{S}}_{n-1}(123,132) \circ (n-1) \cdot \mathcal{S}^*_{n-1}(123,132) & \text{if} \quad n \geq 2. \end{cases} \tag{4}$$

Since φ is an isomorphism, two consecutive permutations in the list (4) differ just by a transposition; so $\mathcal{S}_n(123,132)$ is a Gray code. See Table 1 (a) for $\mathcal{B}_4$ and $\mathcal{S}_5(123,132)$.

Gray code for $S_n(123, 132, \sigma_p)$

The following list is a Gray code for the set $F_n^{(p)}$ [12]:

$$\mathcal{F}_n^{(p)} = \begin{cases} \emptyset & \text{if} \quad n = 0 \\ \{0, 1\} & \text{if} \quad n = 1 \\ 0 \cdot \overline{\mathcal{F}}_{n-1}^{(p)} \circ 10 \cdot \overline{\mathcal{F}}_{n-2}^{(p)} \circ \ldots \circ 1^{p-1}0 \cdot \overline{\mathcal{F}}_{n-p}^{(p)} & \text{if} \quad n > 1 \end{cases} \tag{5}$$

with two conventions: (1) the list $\alpha \cdot \mathcal{F}_{-1}^{(p)}$ consists of the single-string list obtained from α by deleting its last bit, and (2) $\mathcal{F}_{-t}^{(p)}$ is the empty list for $t > 1$. In the list above, two consecutive strings differ in a single position and the listing order is again the dual reflected order. By applying φ, the list $\mathcal{F}_{n-1}^{(p)}$ is transformed into the following list for the set $S_n(123, 132, \sigma_p)$:

$$\mathcal{S}_n(123, 132, \sigma_p) = \begin{cases} \{1\} & \text{if} \quad n = 1 \\ \{21, 12\} & \text{if} \quad n = 2 \\ n \cdot \overline{\mathcal{S}}_{n-1}(123, 132, \sigma_p) & \\ \circ(n-1)n \cdot \overline{\mathcal{S}}_{n-2}(123, 132, \sigma_p) & \\ \ldots & \\ \circ(n-1)(n-2)\ldots(n-p+1)n \cdot \overline{\mathcal{S}}_{n-p}(123, 132, \sigma_p) & \text{if} \quad n > 2. \end{cases} \tag{6}$$

with the conventions: (1) the list $\alpha \cdot \mathcal{S}_0(123, 132, \sigma_p) = \alpha$, and (2) $\mathcal{S}_{-t}(123, 132, \sigma_p)$ is the empty list for $t > 0$.
Since φ is an isomorphism, two consecutive permutations in the list (6) differ just by a transposition; so $\mathcal{S}_n(123, 132, \sigma_p)$ is a Gray code. See Table 1 (b) for $\mathcal{F}_4^{(2)}$ and $\mathcal{S}_5(123, 132, 213)$.

Gray code for $S'_{n,k}(123, 132)$

The following list is the restriction of $\mathcal{B}_n$ defined in (3) to the set $C_{n,k}$. Two consecutive strings differ in two positions and this list is similar to Liu-Tang Gray code [13] except that it lists the strings in dual reflected order.

$$\mathcal{C}_{n,k} = \begin{cases} \emptyset & \text{if } n = 0 \\ \{0^n\} & \text{if } n \geq 1 \text{ and } k = 0 \\ \{1^n\} & \text{if } n \geq 1 \text{ and } k = n \\ 0 \cdot \overline{\mathcal{C}}_{n-1,k} \circ 1 \cdot \mathcal{C}_{n-1,k-1} & \text{if } n \geq 1 \text{ and } 0 < k < n. \end{cases} \tag{7}$$

Under the function φ, $\mathcal{C}_{n-1,k}$ is transformed to the following list for the set $S'_{n,k}(123, 132)$:

$$\mathcal{S}'_{n,k}(123, 132) = \begin{cases} \{1\} & \text{if } n = 1 \\ \{n(n-1)\ldots 21\} & \text{if } n \geq 2 \text{ and } k = 0 \\ \{(n-1)(n-2)\ldots 21n\} & \text{if } n \geq 2 \text{ and } k = n \\ n \cdot \overline{\mathcal{S}}'_{n-1,k}(123, 132) \circ (n-1) \cdot \mathcal{S}'^{*}_{n-1,k-1}(123, 132) & \text{if } n \geq 2 \text{ and } 0 < k < n. \end{cases} \tag{8}$$

Since φ is an isomorphism, two consecutive permutations in the list (8) differ by two transpositions; so $\mathcal{S}'_{n,k}(123, 132)$ is a Gray code. See Table 2 for $\mathcal{C}_{4,2}$ and for $\mathcal{S}'_{5,2}(123, 132)$.

5 Graph Theoretical and Algorithmic Considerations

Here we present some graph-theoretical interpretations of the previous results. The graph *induced* by a combinatorial class has as its vertices the objects of the class, and two vertices of this graph are adjacent if the two corresponding combinatorial objects are close. We denote by $G(X)$ the graph induced by the combinatorial class X and the hypercube Q_n is the graph $G(\{0,1\}^n)$.

Two graphs $G(X)$ and $G(Y)$ are *isomorphic* if there is a bijection $p : X \to Y$ such that two vertices a and b are adjacent in $G(X)$ if and only if the vertices $p(a)$ and $p(b)$ are adjacent in $G(Y)$; so combinatorial and graph isomorphism are equivalent notions; in this case $G(p(X)) = p(G(X))$.

A graph is *connected* if there exists a path between any two vertices. A Hamiltonian path is a path between two vertices of a graph which visits each vertex exactly once. A Hamiltonian path corresponds to a Gray code for the related class.

Figure 2 (a) and (b) show the isomorphic graphs Q_3 and $G(S_4(123,132))$. Hamiltonian paths—or equivalently, Gray codes for the corresponding combinatorial classes—are in thick lines. The graph $G(F_{n-1}^{(p)})$ is the restriction of Q_{n-1} to the set $F_{n-1}^{(p)}$, and in Figure 2 (c) the subgraph $G(F_3^{(2)})$ is in thick lines. Similarly, under the isomorphism $\varphi : F_{n-1}^{(p)} \to S_n(123,132,\sigma_p)$, the graph $G(S_n(123,132,\sigma_p))$ is the restriction of $G(S_n(123,132))$ to the set $S_n(123,132,\sigma_p)$.

For all k, $1 \le k \le n-1$, and under Definition 2, the restriction of Q_{n-1} to the set $C_{n-1,k}$ is not connected because the Hamming distance between any two strings in $C_{n-1,k}$ is at least 2. Now, let G^m be the m-th power of the graph G, i.e., the graph where two vertices are adjacent if there is a path in G of length at most m between these vertices. In this context the graph $G(C_{n-1,k})$ with respect to the closeness Definition 3 is the restriction of Q_{n-1}^2 to the set $C_{n-1,k}$, and under the isomorphism $\varphi : C_{n-1,k} \to S'_{n,k}(123,132)$, $G(S'_{n,k}(123,132))$ is the restriction of $G^2(S_n(123,132))$ to the set $S_{n,k}(123,132)$. In Figure 2 (d), Q_3^2 and its restriction to $C_{3,2}$ are depicted.

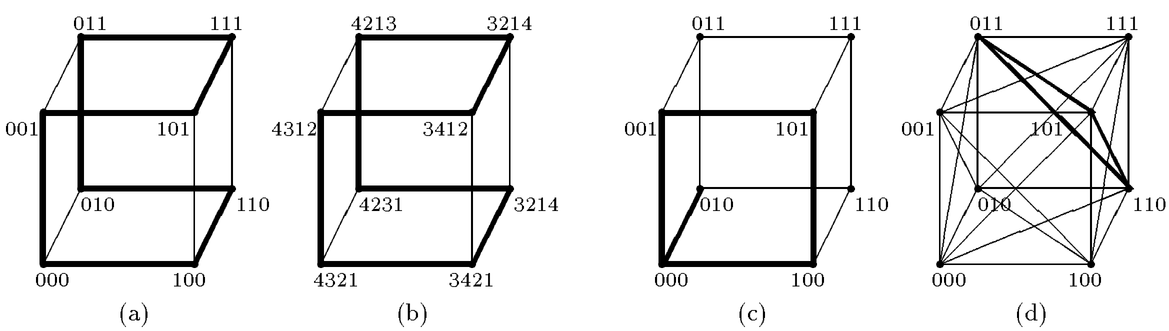


Figure 2: (a) The graph Q_3, and (b) that induced by $S_4(123,132)$; a Hamiltonian cycle in each graph is in thick lines. (c) The graph Q_3 and, in thick lines, its restriction to $F_3^{(2)}$. (d) Q_3^2, the square of the cube, and in thick lines, its restriction to $C_{3,2}$.

Here we show how the isomorphism φ allows us to construct efficient generating algorithms for the lists defined in (4), (6) and (8).

Let b and b' be two successive strings in $\mathcal{B}_n$ defined by (3) and suppose that b and b' differ in position i. Dual-reflected order has the following consequence: either $i = n$ or $b_{i+1} = 0$. Indeed, if $b_{i+1} = 1$, then the string $b_1 \dots b_i 0 b_{i+2} \dots b_n$ is larger than b and smaller than b' in dual reflected order; so b' is not the successor of b. This remark remains true if b and b' are successive strings in $\mathcal{F}_n^{(p)}$. Now let `next` be a procedure which computes the position i where a given string b differs from its successor in the list $\mathcal{X} = \mathcal{B}_{n-1}$ or $\mathcal{X} = \mathcal{F}_{n-1}^{(p)}$. When $\mathcal{X} = \mathcal{B}_{n-1}$, i is alternatively $n-1$ and the rightmost position in b with $b_{i+1} = 0$. When $\mathcal{X} = \mathcal{F}_{n-1}^{(p)}$, `next` is a little more complicated; it is given in [12]. The following algorithm results from the isomorphism φ and generates the list $\mathcal{S}_n(123, 132)$ when $\mathcal{X} = \mathcal{B}_{n-1}$ and the list $\mathcal{S}_n(123, 132, \sigma_p)$ when $\mathcal{X} = \mathcal{F}_{n-1}^{(p)}$.

- Initialize b to the first string in $\mathcal{X}$ and π to $\varphi(b)$. The first string in $\mathcal{B}_{n-1}$ is 01^{n-2} and the first one in $\mathcal{F}_{n-1}^{(p)}$ is given in [12].
- Run `next`. If b differs from its successor in $\mathcal{X}$ in position i, then the successor of π in $\varphi(\mathcal{X})$ is obtained by transposing the entries in position i and $i+1$.
- Stop when the last string in $\mathcal{X}$ is reached. The last string in $\mathcal{B}_{n-1}$ is 1^{n-1} and in the case of the list $\mathcal{F}_{n-1}^{(p)}$ `next` detects its last string in constant time [12].

We now discuss the generation of $S'_{n,k}(123, 132)$ defined in (8), the image of $\mathcal{C}_{n-1,k}$ under the function φ. Let $b = b_1 b_2 \ldots b_n$ be a binary string in $C_{n,k}$ which is not the last one in dual reflected order. Suppose that b differs from its successor, in dual reflected order, in positions i and j, $i < j$. Again, dual-reflected order has the following consequences: (1) either $b_{i+1} = 0$ or $b_{i+1} = b_{i+2} = \ldots = b_n = 1$ (in the latter case $j = i+1$), and (2) either $j = n$ or $b_{j+1} = 0$ or $b_{j+1} = b_{j+2} = \ldots = b_n = 1$. Let `next` be a procedure which computes the positions i and j where a given string b differs from its successor in $\mathcal{C}_{n-1,k}$. Such a procedure can be obtained by a direct implementation of definition (7) or by a slight modification of Liu-Tang algorithm [13]. The following algorithm results from the considerations above and the isomorphism φ; it generates the list $S'_{n,k}(123, 132)$.

- Initialize b to $01^k 0^{n-k-2}$, the first string in $\mathcal{C}_{n-1,k}$, and π to $\varphi(b)$.
- Run `next` and let i and j, $i < j$, be the positions where b differs from its successor in $\mathcal{C}_{n-1,k}$.
  - if $b_{i+1} = 0$, then transpose π_i and π_{i+1}; else transpose π_i and π_n
  - if $j = n-1$ or $b_{j+1} = 0$, then transpose π_j and π_{j+1}; else transpose π_j and π_n.
- Stop when $b = 1^k 0^{n-k-1}$, that is, when the last string in $\mathcal{C}_{n-1,k}$ is reached.

Each of the procedures `next` above has a constant-time implementation and so are the entire generating algorithms.

6 Concluding Remarks

The bijections $\varphi : B_{n-1} \to S_n(123, 132)$, $\varphi : F_{n-1} \to S_n(123, 132, 213)$, and $\varphi : C_{n-1,k} \to S_{n,k}(123, 132)$ are isomorphisms. Since the lists defined in (3), (5), and (7) are Gray codes, so are their images under φ, namely the lists defined by (4), (6), and (8). Tables 1 and 2 are the above-mentioned Gray codes for $n = 5$ (and with $k = 2$ in Table 2).

B_n is a superset of F_n and $C_{n,k}$; also, $S_n(123, 132)$ is a superset of $S_n(123, 132, 213)$ and of $S'_{n,k}(123, 132)$. Our choice of Gray codes (3), (5), and (7) induces some interested properties on their images under φ; the following are two of them.

1. The restriction of the list $\mathcal{B}_{n-1}$ to the set F_{n-1} (resp. $C_{n-1,k}$) is exactly the list $\mathcal{F}_{n-1}$ (resp. $\mathcal{C}_{n-1,k}$), or equivalently $\mathcal{F}_{n-1}$ and $\mathcal{C}_{n-1,k}$ are *(scattered) sublists* of $\mathcal{B}_{n-1}$. For instance, deleting all elements of $\mathcal{B}_4$ having two consecutive 1s from the list in Table 1(a) produces $\mathcal{F}_4$ in Table 1(b); similarly deleting all elements of $\mathcal{B}_4$ having no exactly two 1s from the list in Table 1(a) produces $\mathcal{C}_{4,2}$ in Table 2.
2. The list $\mathcal{S}_n(123, 132, 213)$ and $\mathcal{S}'_{n,k}(123, 132)$ are (scattered) sublists of $\mathcal{S}_n(123, 132)$.

Finally, finding Gray codes for $K_{n-1}^{(p)}$ or, equivalently, for $S_n(123, 132, \sigma_p, T_p)$ remains an open problem.

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