Combinatorial Isomorphism Between Fibonacci Classes

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Abstract

In 1985 Simion and Schmidt showed that the set $S_n(T_3)$ of length n permutations avoiding the set of patterns $T_3 = \{123, 132, 213\}$ is counted by (the second order) Fibonacci numbers. They also presented a constructive bijection between the set F_{n-1} of length $(n-1)$ binary strings with no *two* consecutive 1s and $S_n(T_3)$.

In 2005, Egge and Mansour generalized the first Simion-Simion's result and showed that $S_n(T_p)$, the set of permutations avoiding the patterns $T_p = \{12\dots p, 132, 213\}$, is counted by the $(p-1)$th order Fibonacci numbers.

In this paper we extend the second Simion-Schmidt's result by giving a bijection between the set $F^{(p-1)}_{n-1}$ of length $(n-1)$ binary strings with no $(p-1)$ consecutive 1s, and the set $S_n(T_p)$. Moreover, we show that this bijection is a combinatorial isomorphism, i.e., a closeness-preserving bijection, by which we transform a known Gray code (or equivalently Hamiltonian path) and exhaustive generating algorithm for $F^{(p-1)}_{n-1}$ into similar results for $S_n(T_p)$.

Keywords: Pattern avoiding permutations, generalized Fibonacci strings, Gray codes, combinatorial isomorphism.

1 Introduction and motivation

Let S_ℓ be the set of all permutations of $\{1, 2, \dots, \ell\}$. Let $\pi \in S_n$ and $\tau \in S_k$ be two permutations, $k \le n$. We say that π *contains* τ if there exists a subsequence $1 \le i_1 < i_2 < \dots < i_k \le n$ such that $(\pi_{i_1} \dots \pi_{i_k})$ has all pairwise comparisons the same as τ, i.e., $\tau_s < \tau_t$ whenever $\pi_{i_s} < \pi_{i_t}$ for $1 \le s, t \le k$; in this context τ is usually called a *pattern*. We say that π *avoids* τ, or is τ-*avoiding*, if such a subsequence does not exist. The set of all τ-avoiding permutations in S_n is denoted by $S_n(\tau)$ and $|S_n(\tau)|$ is its cardinality. For an arbitrary finite collection of patterns T, we say that π avoids T if π avoids each $\tau \in T$; the corresponding subset of S_n is denoted by $S_n(T)$ and $|S_n(T)|$ is its cardinality.

The systematic study of pattern avoiding permutations was initiated in 1985 when Simion and Schmidt [8] considered every set of patterns in S_3; two of their propositions are:

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1. For every $n \geq 1$, $|S_n(T_3)| = f_{n+1}$, where $T_3 = \{123, 132, 213\}$ and $\{f_n\}_{n\geq 0}$ is the Fibonacci numbers, initialized by $f_0 = 0$, $f_1 = 1$.

2. For each $n \geq 1$, the following is a constructive bijection between $F_{n-1}^{(2)}$ and the set $S_n(T_3)$ of binary strings of length $(n-1)$ having no *two* consecutive ones: Let $b = b_1 b_2 \ldots b_{n-1} \in F_{n-1}$; its corresponding permutation $\pi \in S_n(T_3)$ is obtained by determining π_i as follows:

   $X^{(1)} = \{1, 2, \ldots, n\}$;
   for $i = 1$ **to** $(n-1)$ **do**
   $$\pi_i = \begin{cases} \text{the largest element in } X^{(i)} \text{ if } b_i = 0 \\ \text{the second largest element in } X^{(i)} \text{ if } b_i = 1 \end{cases}$$
   $X^{(i+1)} = X^{(i)} - \{\pi_i\}$;
   enddo;
   π_n is the unique element in $X^{(n)}$.

We denote this bijection by $\varphi_3 : F_{n-1} \to S_n(T_3)$.

In 2005, Egge and Mansour [1] generalized the first proposition above and showed that for all integers n and $p \geq 2$, $|S_n(T_p)| = f_{n+1}^{(p-1)}$, where $T_p = \{12 \ldots p, 132, 213\}$ and $f_n^{(p)}$ is the n-th Fibonacci number of p-th order (see Section 2 for a definition). This result is a particular case of Mansour's result [6] for the enumeration of $S_n(T, \tau)$ where T consists of two patterns of length three and $\tau \in S_p(T)$.

In the mainstream of the research on pattern-avoiding permutations there is no publication on exhaustive generation nor Gray codes for such permutations, except one of our previous paper [4]. In the present paper, by considering the Egge-Mansour's result, we generalize the second Simion-Schmidt's proposition above and we give a bijection between $F_{n-1}^{(p-1)}$ and $S_n(T_p)$, where $F_n^{(p)}$ is the set of length n binary strings with no p consecutive ones. We also show that this bijection is a combinatorial isomorphism, i.e., a closeness-preserving bijection, by which we construct a Gray code for the set $S_n(T_p)$ which is the image of a known Gray code for $F_{n-1}^{(p-1)}$ [9]. Finally, we give some graph theoretic and algorithmic considerations to illustrate how the concept of combinatorial isomorphism able to translate some properties of its domain to the codomain, and vice versa. A preliminary version of these results were presented in [3] while another approach for a related problem is presented in [4].

2 Generalized Fibonacci Strings

A length n p-th order *Fibonacci string* is a binary string of length n having no p consecutive 1s; the set of such strings is denoted by $F_n^{(p)}$ and it is defined by [9]:

$$F_n^{(p)} = \begin{cases} \{\lambda\} & \text{if} \quad n = 0 \\ \{0, 1\} & \text{if} \quad n = 1 \\ 0 \cdot F_{n-1}^{(p)} \cup 10 \cdot F_{n-2}^{(p)} \cup \ldots \cup 1^{p-1}0 \cdot F_{n-p}^{(p)} & \text{if} \quad n \geq 2 \end{cases} \tag{1}$$

with $p \geq 2$ and λ is the empty string; for an arbitrary binary string α, $\alpha \cdot F_n^{(p)}$ denotes the concatenation of α to every string in $F_n^{(p)}$ with the two following conventions: (1) $\alpha \cdot F_{-1}^{(p)}$ consists of a single element which is α after deleting its last bit, and (2) $\alpha \cdot F_{-t}^{(p)}$ is the empty set for $t > 1$.

It is easy to show that,

$$|F_n^{(p)}| = f_{n+2}^{(p)} \tag{2}$$

where

$$f_m^{(p)} = \sum_{j=1}^{p} f_{m-j}^{(p)} \qquad \text{for } m \geq 2, \tag{3}$$

is the m-th Fibonacci number of p-th order with $f_t^{(p)} = 0$ for $t \leq 0$ and $f_1^{(p)} = 1$ for all p. It is customary to omit the order p in both $F_n^{(p)}$ and $f_n^{(p)}$ when $p = 2$.

3 Bijection $\varphi_p : F_{n-1}^{(p-1)} \to S_n(T_p)$

The Simion-Schmidt bijection $\varphi_3 : F_{n-1} \to S_n(T_3)$, described in the previous section, works entry wise. For $b = b_1 b_2 \ldots b_{n-1} \in F_{n-1}$ let $\varphi_3(b_i)$ denote the i-th entry of $\varphi(b)$. It is easy to show that φ_3 has the following properties:

(i) $\varphi_3(b_i)$ does not return the third largest element in $X^{(i)}$ so the patterns 123 and 132 are avoided,
(ii) since b contains no *two consecutive 1s* therefore there is no i such that $\varphi_3(b_i)$ and $\varphi_3(b_{i+1})$ are both the second largest elements of $X^{(i)}$ and $X^{(i+1)}$, respectively; this ensures the avoidance of the pattern 213.

By considering these properties of φ_3, we construct $\varphi_p : F_{n-1}^{(p-1)} \to S_n(T_p)$, where $T_p = \{12\ldots p, 132, 213\}$, such that φ_p does not return any subsequence containing:

(i) 3-rd largest, largest, and 2-nd largest elements in order to avoid the pattern 132,
(ii) 2-nd largest, 3-rd largest, and largest elements, in order to avoid the pattern 213,
(iii) p consecutive increasing integers, in order to avoid the pattern $12\ldots p$, but at most $(p-1)$ such integers are allowed.

Such φ_p can be formulated as follows: For $b = b_1 b_2 \ldots b_{n-1} \in F_{n-1}^{(p-1)}$ its corresponding permutation $\pi \in S_n(T_p)$ is obtained by determining each π_i as:
$X^{(1)} = \{1, 2, \ldots, n\}$;
for $i = 1$ **to** $(n-1)$ **do**

$$\pi_i = \begin{cases} \text{the largest element in } X^{(i)} \text{ if } b_i = 0 \\ \text{the 2-nd largest element in } X^{(i)} \text{ if } b_i = 1 \text{ and (either } b_{i+1} = 0 \text{ or } i = n-1) \\ \text{the 3-rd largest element in } X^{(i)} \text{ if } b_i = b_{i+1} = 1 \\ \quad \text{and (either } b_{i+2} = 0 \text{ or } i = n-2) \\ \qquad \vdots \\ \text{the } (p-2)\text{-th largest element in } X^{(i)} \text{ if } b_i = b_{i+1} = \ldots = b_{i+p-4} = 1 \\ \quad \text{and (either } b_{i+p-3} = 0 \text{ or } i = n-p+3) \\ \text{the } (p-1)\text{-th largest element in } X^{(i)} \text{ if } b_i = b_{i+1} = \ldots = b_{i+p-3} = 1 \end{cases}$$

$X^{(i+1)} = X^{(i)} - \{\pi_i\}$;
enddo;
π_n is the unique element in $X^{(n)}$.

Proposition 1 *$\varphi_p : F_{n-1}^{(p-1)} \to S_n(T_p)$ is a constructive bijection.*

Proof. This is a consequence of the three considerations above. Indeed, it is easy to verify that φ_p is an injection while $|F_{n-1}^{(p-1)}| = f_{n+1}^{(p-1)}$ (see [1]) and $|S_n(T_p)| = f_{n+1}^{(p-1)}$ (see [4]). $\square$

Example 1

- *The bijection φ_4 maps* 110001 $\in F_6^{(3)}$ *into* (5674312) $\in S_7(T_4)$.
- *The bijection φ_5 between $F_6^{(4)}$ and $S_7(T_5)$ maps* 011101 *into* (7345612)*, and* 011001 *into* (7456312).

Figures 1 and 2 depict these two examples.

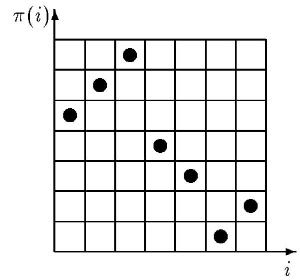

Figure 1: The permutation (5674312) $\in S_7(T_4)$ corresponding to the binary string 110001 $\in F_6^{(3)}$.

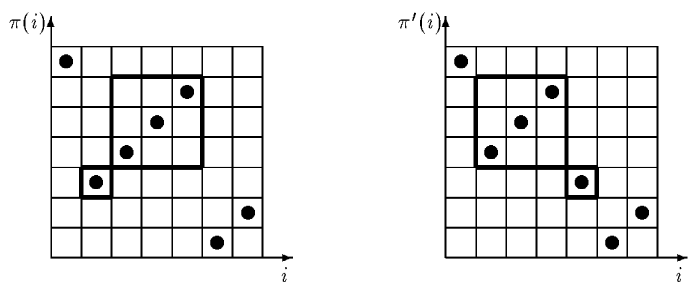

Figure 2: π = (7345612) and π' = (7456312) in $S_7(T_5)$ are the images of 011101 and 011001 in $F_6^{(4)}$, respectively. π' is obtained from π by transposing the left block 3 with the right block 456 in π.

4 Combinatorial isomorphism φ_p

In a permutation, we define a *left block* as a sequence of *increasing consecutive integers* which can not be extended on the left. For instance, consider the permutation $\pi = 56734128 \in S_8$. The sequences 56, 567, 34, 12, and 8 in π are left blocks. Notice that 67 is not a left block since it can be extended on the left as 567. *Right block* is defined similarly. Also notice that 8 is at the same time a left block and a right block.

Definition 1

1. *Two permutations in $S_n(T_p)$ are close if one is obtained from the other by a transposition of two adjacent blocks of total length less than p, one a left block and the other a right;*

2. *Two binary strings are close if they differ in a single position*[1)].

Example 2 *The permutations $\pi = (7\underline{3}\,\underline{456}12)$ and $\pi' = (7\underline{456}\,\underline{3}12)$ in $S_7(T_5)$ are close since π' is obtained from π by transposing the right block 456 with the left block 3 in π; see Figure 2.*

Theorem 1 *The bijection φ_p is a combinatorial isomorphism, that is, two binary strings in $F_{n-1}^{(p-1)}$ are close if and only if their images under this bijection are close in $S_n(T_p)$.*

Proof. Let $b, b' \in F_{n-1}^{(p-1)}$ which differ just in position i, like the following scheme:

$$\begin{array}{l} b = b_1 b_2 \dots b_{t-2} 0 \quad \underbrace{1 \dots 1}_{i-t} b_i \underbrace{1 \dots 1}_{u-i-1} 0 \quad b_{u+1} \dots b_{n-1} \\ b' = b_1 b_2 \dots b_{t-2} 0 \quad \underbrace{1 \dots 1}_{i-t} b'_i \underbrace{1 \dots 1}_{u-i-1} 0 \quad b_{u+1} \dots b_{n-1} \end{array} \tag{4}$$

where $b_t \dots b_{i-1}$ and $b_{i+1} \dots b_{u-1}$ are, possibly empty, contiguous sequences of 1s and $b'_i = 1 - b_i$. Without any loss of generality suppose $b_i = 1$ (and therefore $b'_i = 0$) and in this case $u - t$ (the length of contiguous sequence of 1s , including b_i, in b) is less than or equal to $p - 1$. The shape of π and π', the images of b and b' through the bijection φ_p, are:

$$\begin{array}{rcl} \pi & = & \dots \pi_{t-1} \pi_t \dots \pi_{i-1} \pi_i \pi_{i+1} \dots \pi_u \pi_{u+1} \dots \\ \pi' & = & \dots \pi_{t-1} \pi'_t \dots \pi'_{i-1} \pi'_i \pi'_{i+1} \dots \pi'_u \pi_{u+1} \dots . \end{array} \tag{5}$$

Note that $\pi_t \dots \pi_i \dots \pi_u$ is at once a left and a right block in π and so are $\pi'_t \dots \pi'_i$ and $\pi'_{i+1} \dots \pi'_u$ in π'. Since $\{\pi_t, \dots, \pi_i, \dots, \pi_u\}$ and $\{\pi'_t, \dots, \pi'_i, \dots, \pi'_u\}$ are equal (as sets, but different as sequences) and $\pi'_u < \pi'_t$ (actually $\pi'_u = \pi'_t - 1$) we have $\pi_t \dots \pi_i \pi_{i+1} \dots \pi_u = \pi'_{i+1} \dots \pi'_u \pi'_t \dots \pi'_i$. □

5 Gray code for $S_n(T_p)$

In this section we show how a combinatorial isomorphism transforms a known Gray code for Fibonacci strings into a Gray code for the set of permutations $S_n(T_p)$.

By definition, a Gray code for a combinatorial family is a listing of objects in the family such that successive objects differ in some pre-specified, usually small, way [2]. In [9] a Gray code list for the set of Fibonacci strings defined by (1) is given. In this list successive strings differ in a single position and its definition is:

$$\mathcal{F}_n^{(p)} = \begin{cases} \lambda & \text{if} \quad n = 0 \\ 0, 1 & \text{if} \quad n = 1 \\ 0 \cdot \overline{\mathcal{F}}_{n-1}^{(p)} \circ 10 \cdot \overline{\mathcal{F}}_{n-2}^{(p)} \circ \dots \circ 1^{p-1} 0 \cdot \overline{\mathcal{F}}_{n-p}^{(p)} & \text{if} \quad n > 1 \end{cases} \tag{6}$$

[1)] See [7, pp.116] for general setting on closeness relations.

Table 1: (a) The list $\mathcal{F}_5^{(2)}$ and its image $\mathcal{S}_6(T_3) = \mathcal{S}_6(123, 132, 213)$, and (b) The list $\mathcal{F}_4^{(3)}$ and its image $\mathcal{S}_5(T_4) = \mathcal{S}_5(1234, 132, 213)$ together with the Hamming distances between consecutive permutations. Note that the Hamming distance between any two consecutive elements of $\mathcal{S}_6(T_3)$ is two.

(a)		(b)		
$\mathcal{F}_5^{(2)}$	$\mathcal{S}_6(T_3)$	$\mathcal{F}_4^{(3)}$	$\mathcal{S}_5(T_4)$	distance
01001	645312	0110	52341	
01000	645321	0100	53421	3
01010	645231	0101	53412	2
00010	654231	0001	54312	2
00000	654321	0000	54321	2
00001	654312	0010	54231	2
00101	653412	0011	54123	3
00100	653421	1011	45123	2
10100	563421	1010	45231	3
10101	563412	1000	45321	2
10001	564312	1001	45312	2
10000	564321	1101	34512	3
10010	564231	1100	34521	2

where $\circ$ is the operator of concatenation of two lists, $\overline{\mathcal{F}}$ is the list obtained by reversing $\mathcal{F}$, and with two conventions: (1) the list $\alpha \cdot \mathcal{F}_{-1}^{(p)}$ consists of the single string list obtained from α by deleting its last bit, and (2) $\mathcal{F}_{-t}^{(p)}$ is the empty list for $t > 1$.

By applying the combinatorial isomorphism φ_p to each binary string in the list $\mathcal{F}_n^{(p)}$ one obtains a list for the set $S_{n+1}(T_{p+1})$; or equivalently, by the combinatorial isomorphism φ_p, the Gray code $\mathcal{F}_{n-1}^{(p-1)}$ is transformed into the list $\mathcal{S}_n(T_p)$ for the set $S_n(T_p)$ defined by:

$$\mathcal{S}_n(T_p) = \begin{cases} (1) & \text{if} \quad n = 1 \\ (21), (12) & \text{if} \quad n = 2 \\ n \cdot \overline{\mathcal{S}}_{n-1}(T_p) \circ (n-1)n \cdot \overline{\mathcal{S}}_{n-2}(T_p) \circ \ldots & \\ \circ (n-p+2) \ldots n \cdot \overline{\mathcal{S}}_{n-p+1}(T_p) & \text{if} \quad n > 2 \end{cases} \tag{7}$$

with the conventions: (1) the list $\alpha \cdot \mathcal{S}_0(T_p) = \alpha$, and (2) $\mathcal{S}_{-t}(T_p)$ is the empty list for $t > 0$. Table 1 shows the lists $\mathcal{F}_5^{(2)}$ and $\mathcal{F}_4^{(3)}$ with their images $\mathcal{S}_6(T_3)$ and $\mathcal{S}_5(T_4)$, respectively.

Since any two consecutive strings in $\mathcal{F}_{n-1}^{(p-1)}$ are close, by Theorem 1, so are their images through the combinatorial isomorphism φ_p, hence the Hamming distance between consecutive permutations in $\mathcal{S}_n(T_p)$ is less than p. The following lemma formalizes this result using a different approach from Theorem 1.

Lemma 1 *The Hamming distance between any two consecutive elements of $\mathcal{S}_n(T_p)$ is upper bounded by the minimum between $(p-1)$ and n.*

Proof. We consider p fixed. Obviously, the Hamming distance between two consecutive elements in $\mathcal{S}_n(T_p)$ is less than or equal to n. Suppose $n \geq p$ and let

$$(n-k)(n-k+1)\dots n \cdot \overline{\mathcal{S}}_{n-k-1}(T_p) \tag{8}$$

and

$$(n-k-1)(n-k)\dots n \cdot \overline{\mathcal{S}}_{n-k-2}(T_p) \tag{9}$$

be two (not empty) consecutive sublists in the definition (7), for $n > 2$, with $0 \leq k \leq p-3$. We show that the Hamming distance between the last element in the list (8) and the first one in (9) is less than p.

$$\begin{aligned}
&\mathit{last}\left((n-k)(n-k+1)\dots n \cdot \overline{\mathcal{S}}_{n-k-1}(T_p)\right)\\
&= (n-k)(n-k+1)\dots n \cdot \mathit{last}(\overline{\mathcal{S}}_{n-k-1})(T_p)\\
&= (n-k)(n-k+1)\dots n \cdot \mathit{first}(\mathcal{S}_{n-k-1})(T_p)\\
&= (n-k)(n-k+1)\dots n(n-k-1) \cdot \mathit{first}(\overline{\mathcal{S}}_{n-k-2}(T_p)).
\end{aligned}$$

So, the last element in the list defined by (8) differs from the first element in the list defined by (9) in exactly $k+2 \leq p-1$ positions. Induction on n completes the proof. $\square$

The following remark is useful for the generating algorithm sketched in the last section.

Remark 1 *Let b and b' be two binary strings in the list $\mathcal{F}_{n-1}^{(p-1)}$ with b' the successor of b and π and π' their images by bijection φ_p as in schemes (4) and (5). If b differs from b' in position i then either $i = n-1$ or $b_{i+1} = b'_{i+1} = 0$, see [9]. With the notations in the proof of Theorem 1*

- *if $b_i = 0$ then π' is obtained from π by transposing the block $\pi_t \dots \pi_i$ (with $t = i$ if $i = 1$ or $b_{i-1} = 0$) with the single element block π_{i+1},*
- *if $b_i = 1$ then π' is obtained from π by transposing the single element block π_t with the block $\pi_{t+1} \dots \pi_{i+1}$.*

See Figure 2.

Note that when $p = 3$ consecutive permutations in $\mathcal{S}_n(T_3)$ differ by the transposition of two adjacent elements; see Table 1(a).

6 Graph theoretic and algorithmic considerations

The isomorphism shown by Theorem 1 also has a graph theoretical meaning. Let X be a class of combinatorial objects and $G(X)$ be the graph induced by X, i.e., the graph with vertex set X, and edges connecting *close* vertices. With this terminology, a Gray code for X is a Hamiltonian path for $G(X)$.

Theorem 1 implies that the bijection φ_p is a graph isomorphism between $G(F_{n-1}^{(p-1)})$ and $G(S_n(T_p))$; this isomorphism transforms the Hamiltonian path $\mathcal{F}_{n-1}^{(p-1)}$ defined by (6) into the Hamiltonian path $\mathcal{S}_n(T_p)$ defined by (7). Figure 3 shows the graphs $G(F_4^{(3)})$ and $G(S_5(T_4))$ where the Hamiltonian paths $\mathcal{F}_4^{(3)}$ and $\mathcal{S}_5(T_4)$ are in bold.

Now, we explain how a slight modification of an efficient exhaustive generation algorithm for the list $\mathcal{F}_{n-1}^{(p-1)}$ transforms it into a similar algorithm for $\mathcal{S}_n(T_p)$. In [9] is presented the

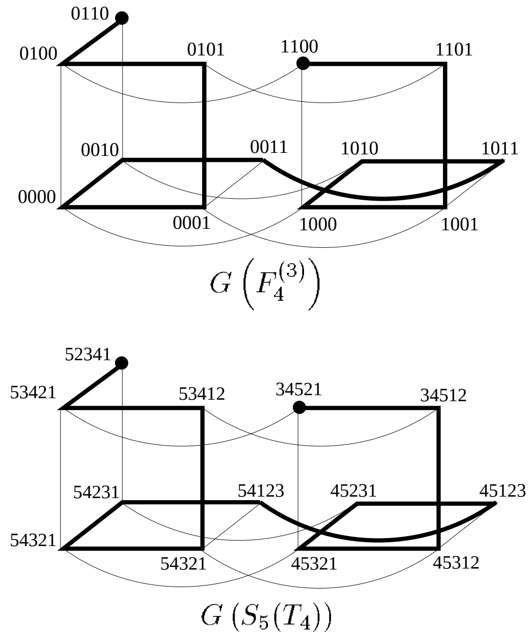

Figure 3: The isomorphic graphs $F_4^{(3)}$ and $S_5(T_4)$. Two vertices in $F_4^{(3)}$ are connected if their Hamming distance is one, while two vertices in $S_5(T_4)$ are connected if one is obtained from the other by transposing two adjacent blocks of size at most three. Bold lines are the Hamiltonian paths listed in Table 1 (b).

loopless procedure *next* which after a linear-time precomputation step (and using additional data structures) computes, in constant time, the position i where the current string belonging to $\mathcal{F}_{n-1}^{(p-1)}$ must be changed in order to obtain the next one. *next* subsequently computes the length of the contiguous sequence of 1s ending in position $i-1$ (that is, $i-t$ with the notations in the proof of Theorem 1).

The following scheme yields a generating algorithm for $\mathcal{S}_n(T_p)$. Initialize b by the first string in $\mathcal{F}_{n-1}^{(p-1)}$ as in [9] and π by its image through the bijection φ_p; then, run *next* and update π as in Remark 1. The time complexity of the obtained algorithm is given by the second step—the blocks transposition— and it is $O(p)$ per permutation, independent of n. A linked representation for π can reduce this complexity to $O(1)$; see [5] for a detailed explanation of this technique.

References

[1] E.S. Egge and T. Mansour. *Restricted Permutations, Fibonacci Numbers, and k-generalized Fibonacci Numbers.* Integers: Electronic Journal of Cominatorial Number Theory **5**(1) : 12pp, 2005.

[2] J.T. Joichi, D.E. White, and S.G. Williamson. *Combinatorial Gray Codes.* SIAM Journal on Computing, **9**(1) : 130-141, 1980.

[3] A. Juarna and V. Vajnovszki. *Isomorphism Between Classes Counted by Fibonacci Numbers.* Proceeding of Words 2005 : 51-62, Montreal-Canada, September 2005.

[4] A. Juarna and V. Vajnovszki. *Combinatorial isomorphisms beyond a Simion-Schmidt's bijection.* Proceeding of CANT 2006 : 11pp, Liège-Belgique, May 2006.

[5] J.F. Korsh and S. Lipschutz. *Generating multiset permutations in constant time.* J. Algorithms, **25**(1) : 321-335, 1997.

[6] T. Mansour. *Permutations avoiding pattern from S_k and at least two patterns from S_3.* Ars Combinatorica **62** : 227-239, 2001.

[7] F. Ruskey. *Combinatorial Generation*, textbook for CSC-425/520 Univ. Victoria. (Available at http://www.1stworks.com/ref/RuskeyCombGen.pdf)

[8] R. Simion and F.W. Schmidt. *Restricted Permutations.* Europ. J. Combinatorics, **6** : 383-406, 1985.

[9] V. Vajnovszki. *A Loopless Generation of Bitstring without p Consecutive Ones.* Proceeding of 3-rd Discrete Mathematics and Theoretical Computer Science, 227-239, 2001.